SOLUTIONS MANUAL

THIRD EDITION

MECHANICAL VIBRATIONS

SINGIRESU S. RAO

PURDUE UNIVERSITY

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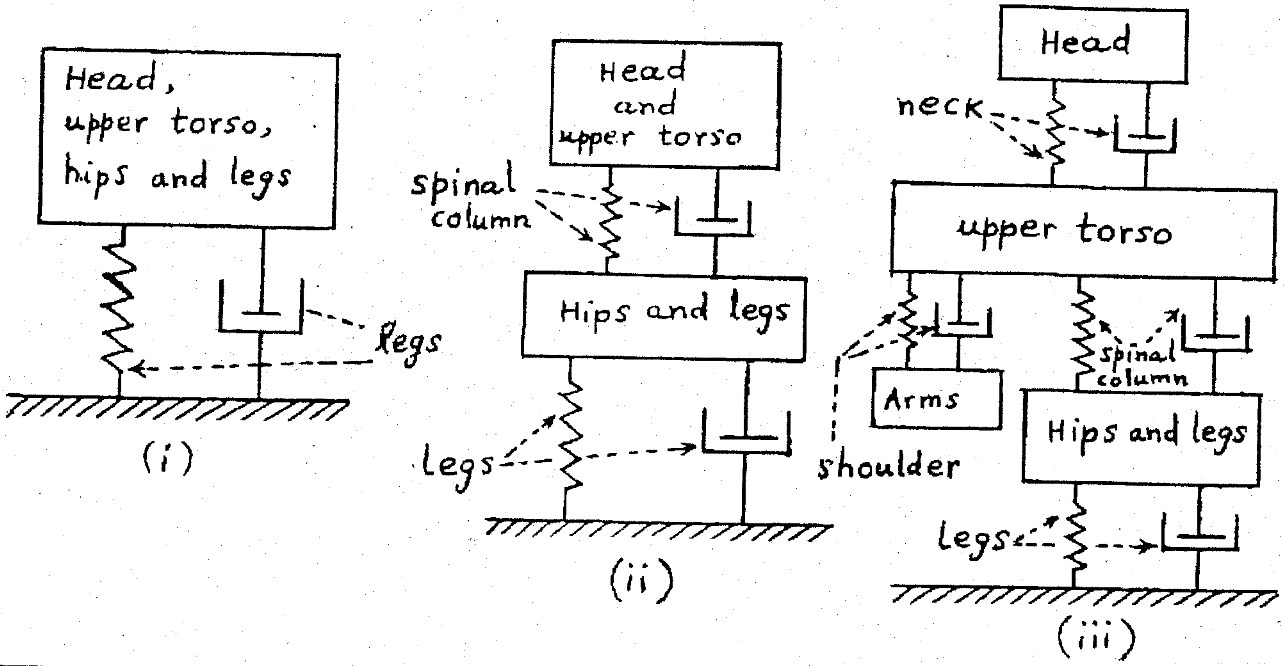
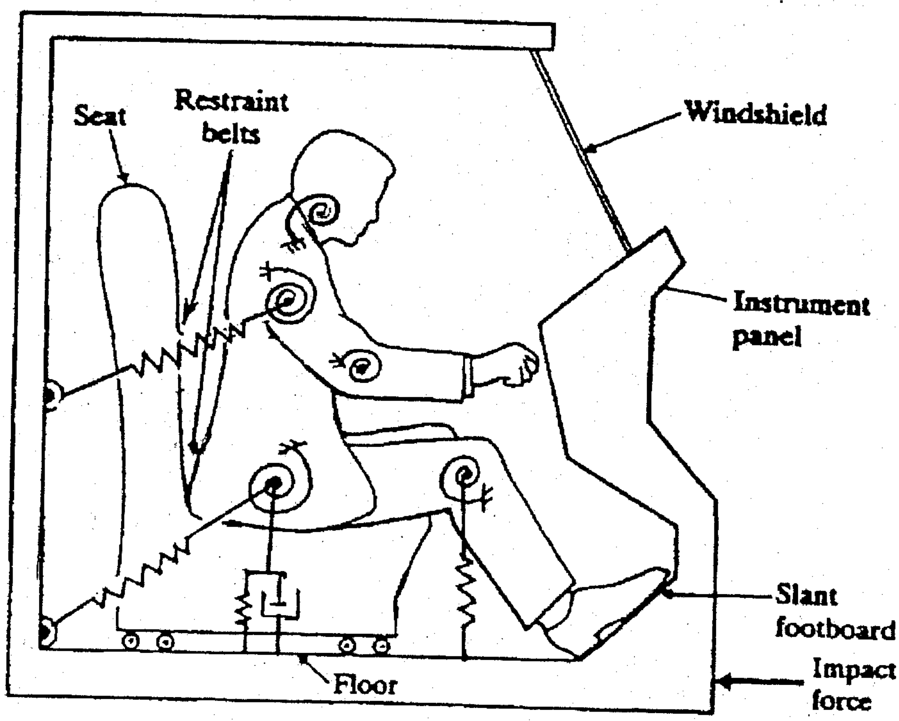
Chapter 1

Fundamentals of Vibration

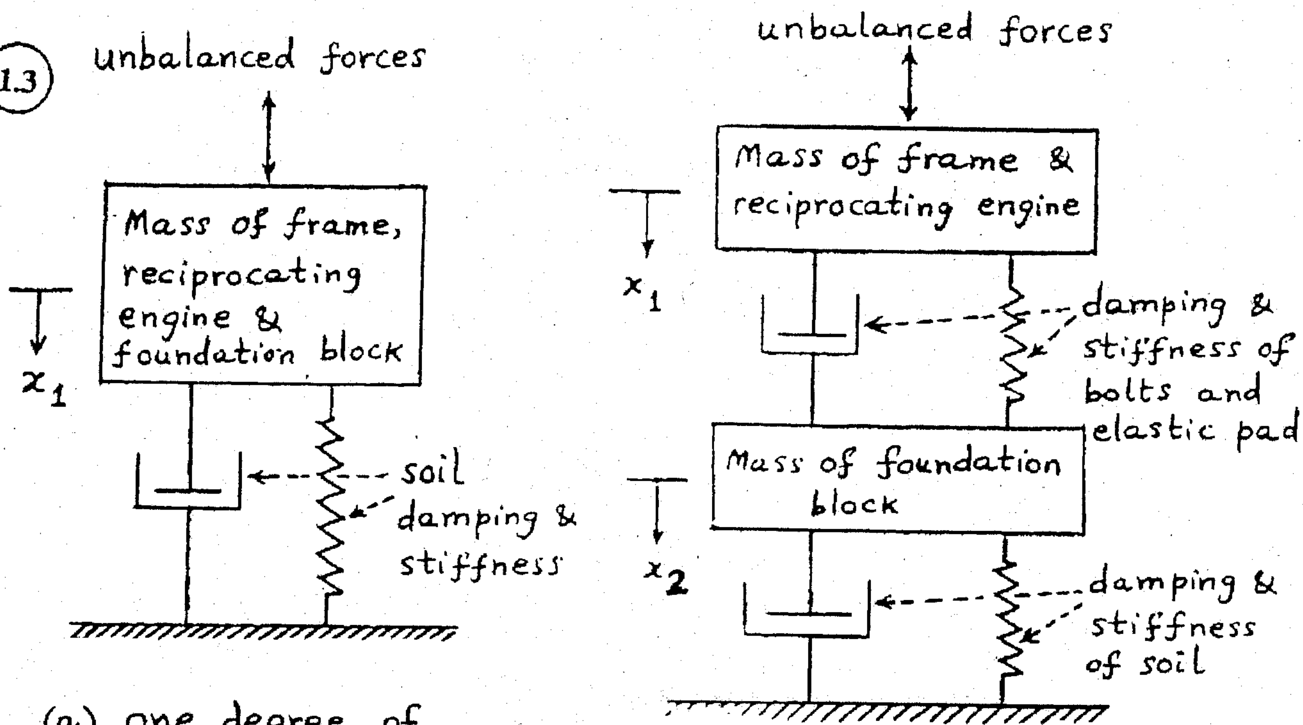
1.1



1.2



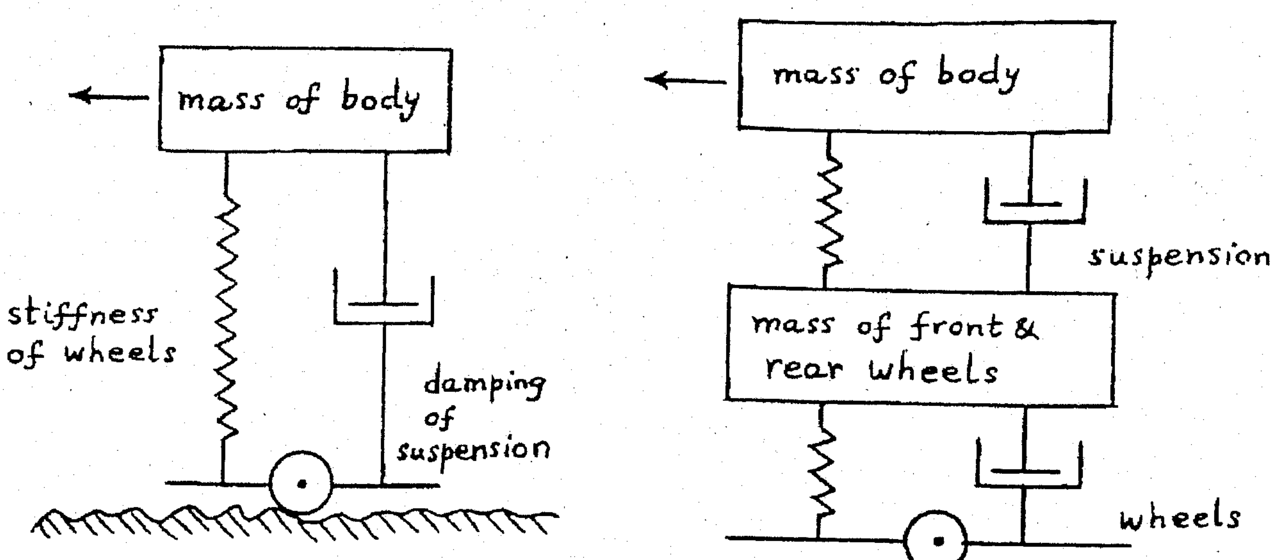
1.3



(a) one degree of freedom model

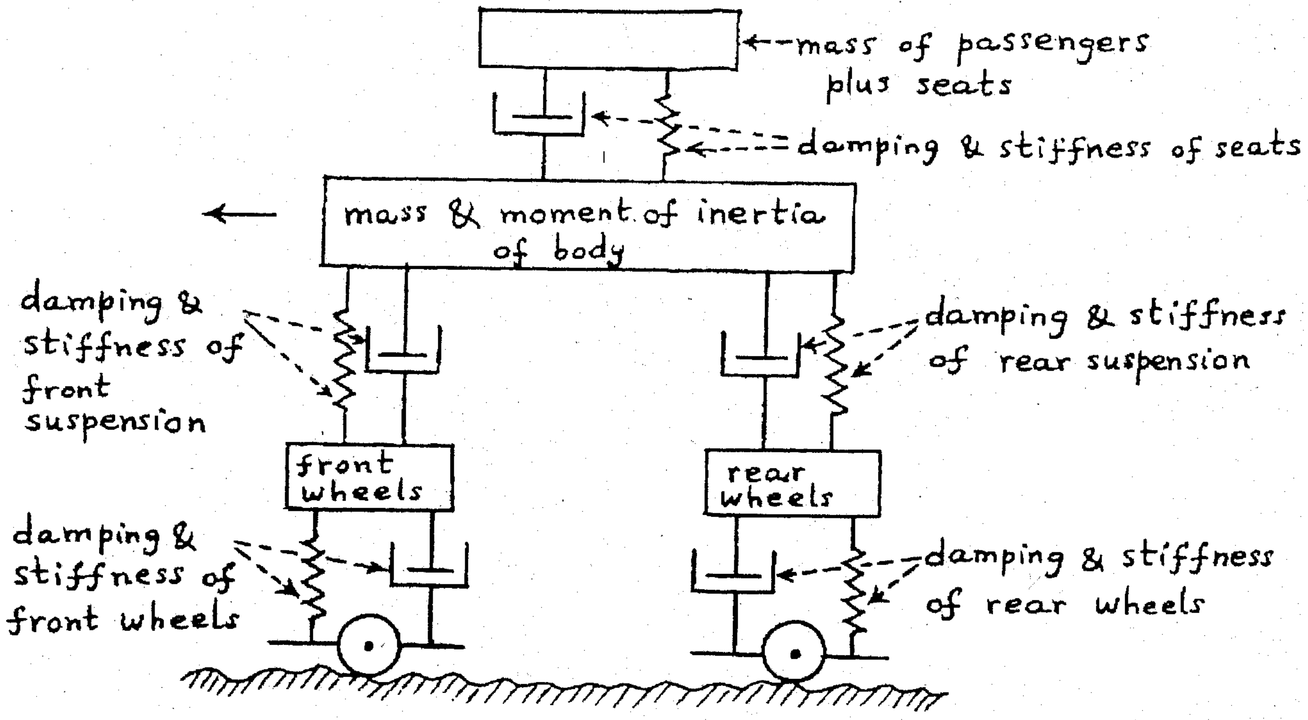
(b) Two degree of freedom model

1.4



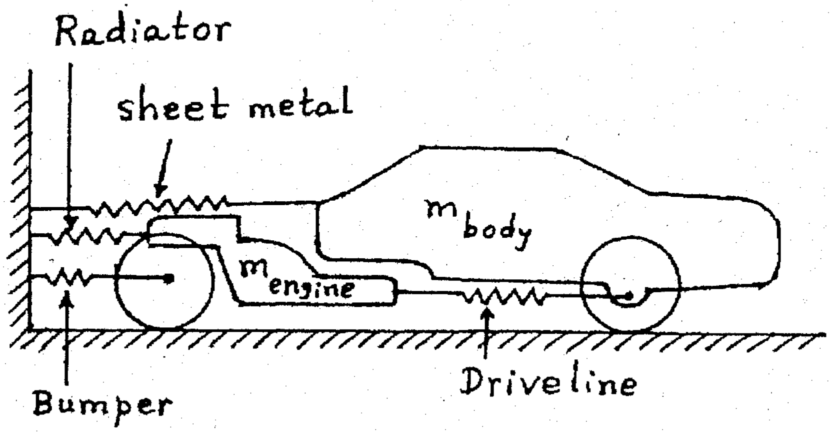
(i)

(ii)

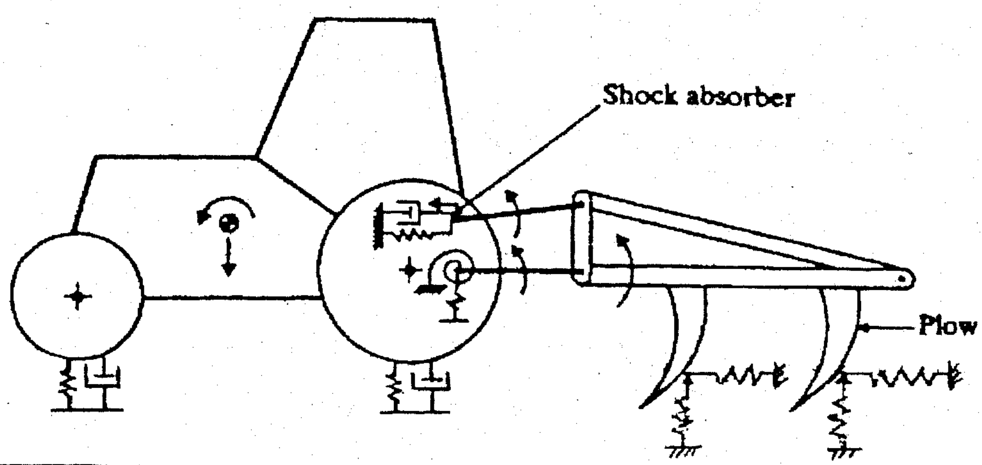

mass of passengers plus seats
damping & stiffness of seats
mass & moment of inertia of body
damping & stiffness of front suspension
damping & stiffness of rear suspension
front wheels
rear wheels
damping & stiffness of front wheels
damping & stiffness of rear wheels

(iii)

(1.5)

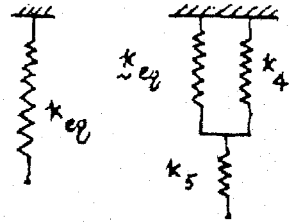

Radiator
sheet metal
m_{body}
m_{engine}
Bumper
Driveline

(1.6)


Shock absorber
Plow

1.7

$$\frac{1}{\tilde{k}_{eq}} = \frac{1}{2k_1} + \frac{1}{k_2} + \frac{1}{2k_3} \quad ; \quad \tilde{k}_{eq} = \left(\frac{2k_1k_2k_3}{k_2k_3 + 2k_1k_3 + k_1k_2} \right)$$



$$\frac{1}{k_{eq}} = \frac{1}{\tilde{k}_{eq} + k_4} + \frac{1}{k_5}$$

$$k_{eq} = \frac{k_5(\tilde{k}_{eq} + k_4)}{k_5 + k_4 + \tilde{k}_{eq}} = \frac{k_2k_3k_4k_5 + 2k_1k_3k_4k_5 + k_1k_2k_4k_5 + 2k_1k_2k_3k_5}{k_2k_3k_4 + k_2k_3k_5 + 2k_1k_3k_4 + 2k_1k_3k_5 + k_1k_2k_4 + k_1k_2k_5 + 2k_1k_2k_3}$$

1.8 Equivalence of potential energies gives

$$\frac{1}{2}k_{t1}\theta^2 + \frac{1}{2}k_{t2}\theta^2 + \frac{1}{2}k_1(\theta l_1)^2 + \frac{1}{2}k_2(\theta l_1)^2 + \frac{1}{2}k_3(\theta l_2)^2 = \frac{1}{2}k_{eq}\theta^2$$

$$\therefore \quad k_{eq} = k_{t1} + k_{t2} + k_1 l_1^2 + k_2 l_1^2 + k_3 l_2^2$$

1.9 k_{123} = for series springs k_1, k_2 and k_3:

$$\frac{1}{k_{123}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \quad ; \quad k_{123} = \frac{k_1k_2k_3}{k_1k_2 + k_2k_3 + k_3k_1}$$

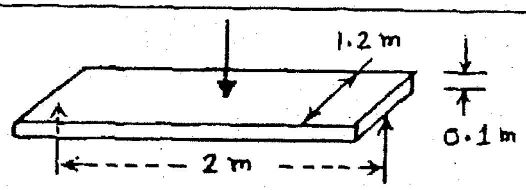
Using energy equivalence,

$$\frac{1}{2}k_{eq}\theta^2 = \frac{1}{2}k_4\theta^2 + \frac{1}{2}k_{123}\theta^2 + \frac{1}{2}k_5(\theta R)^2 + \frac{1}{2}k_6(\theta R)^2$$

$$\therefore \; k_{eq} = k_4 + k_{123} + R^2 k_5 + R^2 k_6$$

$$= k_4 + \left(\frac{k_1k_2k_3}{k_1k_2 + k_2k_3 + k_3k_1} \right) + R^2(k_5 + k_6)$$

1.10 For simply supported beam, for load at middle,



$$k_1 = \frac{48EI}{l^3} = \frac{48(2.06 \times 10^{11})(10^{-4})}{8}$$

$= 12.36 \times 10^7$ N/m where $I = \frac{1}{12}(1.2)(0.1)^3 = 10^{-4}\ m^4$.

δ_1 = original deflection $= \frac{mg}{k_1} = \frac{500 \times 9.81}{12.36 \times 10^7} = 396.8447 \times 10^{-7}$ m

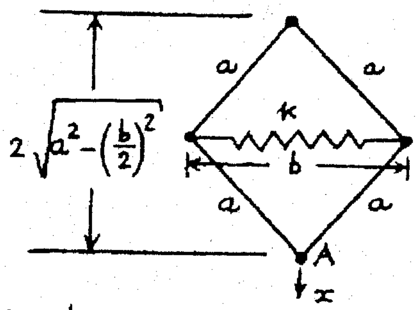
When spring k is added, $k_{eq} = k + k_1$

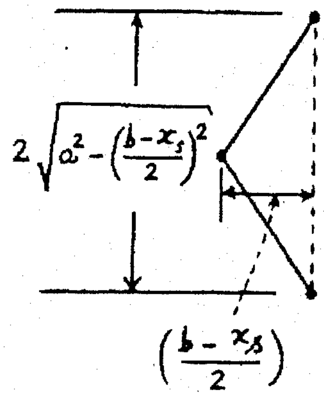
New deflection $= \frac{mg}{k_{eq}} = \frac{\delta_1}{3}$; $k_{eq} = \frac{3\,mg}{\delta_1} = 3k_1$

$= k + k_1$

$k = 2k_1 = 24.72 \times 10^7$ N/m

(1.11) (a) x = downward deflection of point A,
x_s = resulting deformation of spring





Potential energy equivalence gives $\frac{1}{2}k_{eq}x^2 = \frac{1}{2}k\,x_s^2$

$$k_{eq} = k\left(\frac{x_s}{x}\right)^2$$

But
$$x = 2\left[\sqrt{a^2-\left(\frac{b-x_s}{2}\right)^2} - \sqrt{a^2-\left(\frac{b}{2}\right)^2}\right]$$
$$= 2\sqrt{a^2-\left(\frac{b}{2}\right)^2}\left[\left\{\frac{a^2-\left\{\frac{b}{2}\left(1-\frac{x_s}{b}\right)\right\}^2}{a^2-\left(\frac{b}{2}\right)^2}\right\}^{1/2} - 1\right]$$
$$= 2\sqrt{a^2-\frac{b^2}{4}}\left[\left\{\frac{\left(a^2-\frac{b^2}{4}-\frac{x_s^2}{4}+\frac{b\,x_s}{2}\right)}{\left(a^2-\frac{b^2}{4}\right)}\right\}^{1/2} - 1\right]$$
$$= 2\sqrt{a^2-\frac{b^2}{4}}\left[\left\{1-\frac{\overset{\approx 0}{x_s^2}}{\left(a^2-\frac{b^2}{4}\right)}+\frac{b\,x_s}{2\left(a^2-\frac{b^2}{4}\right)}\right\}^{1/2} - 1\right]$$

Using the relation $(1+\theta)^{1/2} \simeq 1+\frac{\theta}{2}$, we obtain

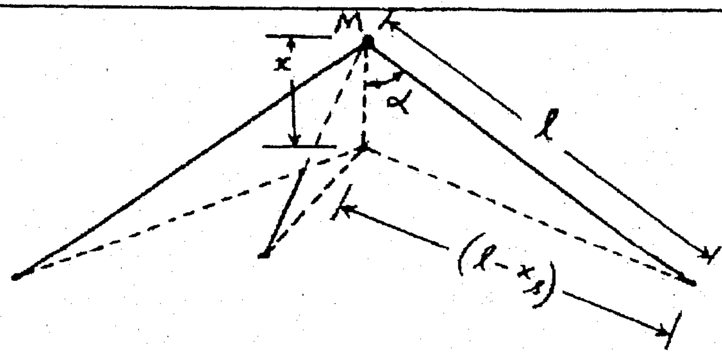
$$x = 2\left(a^2-\frac{b^2}{4}\right)^{1/2}\left[1+\frac{b\,x_s}{4\left(a^2-\frac{b^2}{4}\right)} - 1\right] = \frac{b\,x_s}{2\left(a^2-\frac{b^2}{4}\right)^{1/2}}$$

$$\therefore\; k_{eq} = k\left(\frac{x_s}{x}\right)^2 = 4k\left(\frac{a^2-\frac{b^2}{4}}{b^2}\right) = k\left(\frac{4a^2-b^2}{b^2}\right)$$

(b) Here $x = x_s$ (spring deflection)

$$\therefore\; k_{eq} = k$$

(1.12) Let x = vertical displacement of mass M;
x_s = resulting deformation of each inclined spring.



From equivalence of potential energy,

$$\frac{1}{2} k_{eq}\, x^2 = 3\left(\frac{1}{2} k\, x_s^2\right) \quad ; \quad k_{eq} = 3k\left(\frac{x_s}{x}\right)^2$$

From geometry,

$$(l - x_s)^2 = l^2 + x^2 - 2 l x \cos\alpha$$

$$x^2 - 2x\, l \cos\alpha + 2 l\, x_s - x_s^2 = 0 \qquad (E_1)$$

Solving (E_1),

$$x = l\cos\alpha\left[1 \pm \left\{1 - \frac{(2 l x_s - x_s^2)}{l^2 \cos^2\alpha}\right\}^{1/2}\right] \qquad (E_2)$$

Using the relation $\sqrt{1-\theta} \approx 1 - \frac{\theta}{2}$, (E_2) can be rewritten as

$$x = l\cos\alpha\left[1 \pm \left\{1 - \left(\frac{2 l x_s - x_s^2}{2 l^2 \cos^2\alpha}\right)\right\}\right] \qquad (E_3)$$

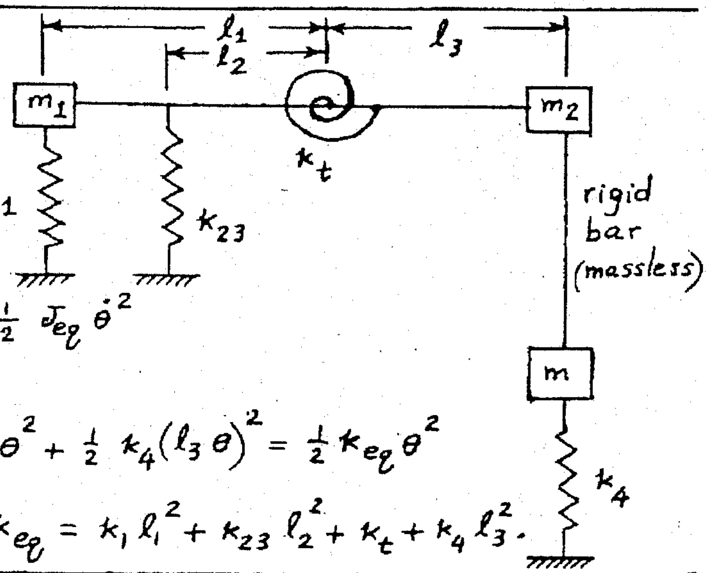
Assuming x to be small, we use minus sign and neglect x_s^2 compared to $2 l x_s$ in (E_3). This gives

$$x = \frac{x_s}{\cos\alpha}$$

$$\therefore \quad k_{eq} = 3k\cos^2\alpha$$

In a similar manner, $c_{eq} = 3c\cos^2\alpha$

1.13



$$k_{23} = \frac{k_2 k_3}{k_2 + k_3}$$

$$k_4 = A\rho g = \frac{\pi d^2}{4}\rho g$$

From kinetic energy,

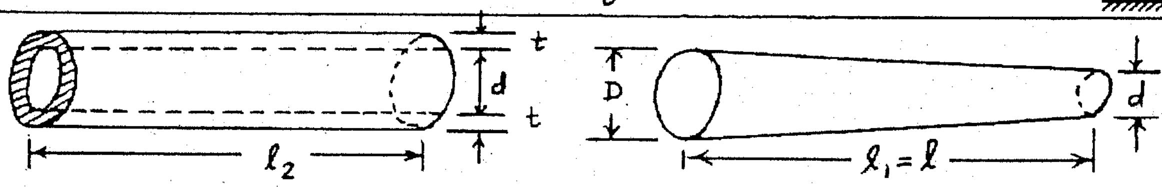
$$\frac{1}{2} m_1 (l_1\dot\theta)^2 + \frac{1}{2}(m_2 + m)(l_3\dot\theta)^2 = \frac{1}{2} J_{eq}\,\dot\theta^2$$

From potential energy,

$$\frac{1}{2} k_1 (l_1\theta)^2 + \frac{1}{2} k_{23}(l_2\theta)^2 + \frac{1}{2} k_t\theta^2 + \frac{1}{2} k_4 (l_3\theta)^2 = \frac{1}{2} k_{eq}\theta^2$$

$$\therefore \quad J_{eq} = m_1 l_1^2 + (m_2 + m)\, l_3^2 \quad ; \quad k_{eq} = k_1 l_1^2 + k_{23} l_2^2 + k_t + k_4 l_3^2 .$$

1.14



$$k_2 = \frac{EA}{l_2} = \frac{\pi E t (d+t)}{l_2} \qquad k_1 = \frac{\pi E D d}{4 l}$$

$k_2 = k_1$ gives $\quad l_2 = \dfrac{4 t (d+t)}{D d}$

(1.15) (a) $F \simeq F\Big|_{x_0} + \frac{dF}{dx}\Big|_{x_0}(x - x_0) = \left(500x + 2x^3\right)_{x=10} + \left(500 + 6x^2\right)_{x=10} \cdot (x-10)$

$\simeq 1100\,x - 4000$

(b) at $x = 9$ mm:

Exact $F_9 = 500 \times 9 + 2\,(9)^3 = 5958$ N

Approximate $F_9 = 1100 \times 9 - 4000 = 5900$ N

Error $= -0.9735\,\%$

(c) at $x = 11$ mm:

Exact $F_{11} = 500 \times 11 + 2\,(11)^3 = 8162$ N

Approximate $F_{11} = 1100 \times 11 - 4000 = 8100$ N

Error $= +0.7596\,\%$

(1.16) $p v^{\gamma} = \text{constant}$ --- (E_1) ; Differentiation of (E_1) gives

$$dp\; v^{\gamma} + p\,\gamma\, v^{\gamma-1}\, dv = 0$$

$$dp = -\frac{p\gamma}{v}\, dv \quad \text{---} (E_2)$$

change in volume when mass moves by dx, $dv = -A \cdot dx$ --- (E_3)

Eqs. (E_2) and (E_3) give $dp = \frac{p\gamma A}{v}\, dx$

Force due to pressure change $= dF = dp \cdot A = \frac{p\gamma A^2}{v} \cdot dx$

spring constant of air spring $= k = \frac{dF}{dx} = \left(\frac{p\gamma A^2}{v}\right)$.

(1.17) Equivalent spring constants in diffent directions are

$k_{e1} = \left(\frac{k_5 k_6 k_7}{k_5 k_6 + k_5 k_7 + k_6 k_7}\right)$, $\quad k_{e2} = \left(\frac{k_8 k_9}{k_8 + k_9}\right)$.

$k_{e3} = \left(\frac{k_1 k_2}{k_1 + k_2}\right)$, $\quad k_{e4} = \left(\frac{k_3 k_4}{k_3 + k_4}\right)$

If the force P moves by x, spring located at θ_i undergoes a displacement of $x_i = x \cos\theta_i$ (derivation as in problem 1.6)

Equivalence of potential energy gives $\frac{1}{2} k_{eq}\, x^2 = \frac{1}{2} \sum_{i=1}^{4} k_{ei}\, x_i^2$

$$k_{eq} = \sum_{i=1}^{4} \left(k_{ei} \cos^2\theta_i\right)$$

(1.18) From Problem 1.16, $k = \frac{p\gamma A^2}{v}$ with $\gamma = 1.4$ for air

Let $p = 200$ psi

$k = 75$ lb/in $= \frac{(200)(1.4)\,A^2}{v} \Rightarrow \frac{A^2}{v} = 0.2679$

Let diameter of piston $= d = 2$ inch ; $A = \frac{\pi}{4}(2)^2 = 3.1416$ in^2

$v = A^2/0.2679 = 36.8408 \text{ in}^3$

Let $h = 2$ inch ; $\frac{\pi}{4} D^2 (2) = v \Rightarrow D = 4.8429$ inch

---

1.19 $F = a\,x + b\,x^3 = 2\,(10^4)\,x + 4\,(10^7)\,x^3$

Around x^*: $F(x) \approx F(x^*) + \frac{dF}{dx}\big|_{x^*} (x - x^*)$

When $x^* = 10^{-2}$ m, $F(x^*) = 2\,(10^4)\,(10^{-2}) + 4\,(10^7)\,(10^{-6}) = 240$ N

$\frac{dF}{dx}\big|_{x^*} = a + 3\,b\,x^2 = 2\,(10^4) + 3\,(4)\,(10^7)\,(10^{-4}) = 32000$

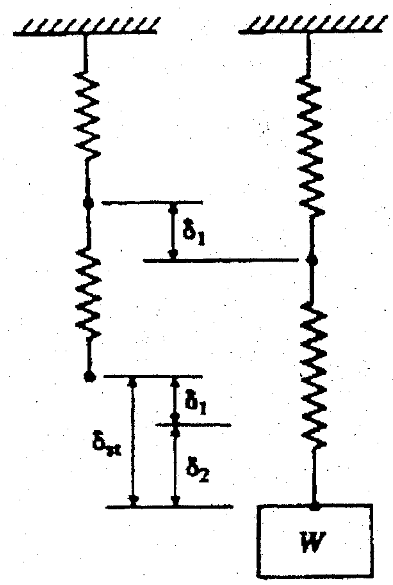
Hence $F(x) = 240 + 32000\,(x - 0.01) = (32000\,x - 80)$ N
Since the linearized spring constant is given by $F(x) = k_{eq}\,x$, we have $k_{eq} = 32{,}000$ N/m.

1.20 $F_i = a_i\,x_i + b_i\,x_i^3$; $i = 1, 2$

Springs in series:

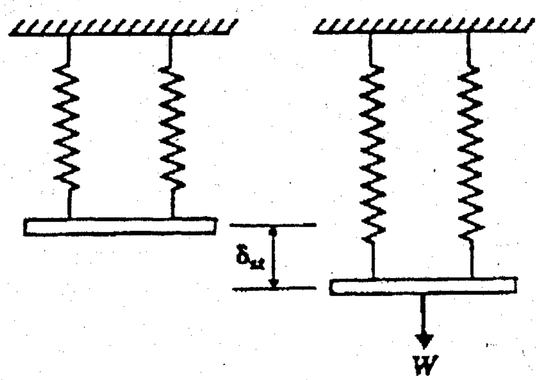
$$W = a_1\,\delta_1 + b_1\,\delta_1^3 \qquad (1)$$
$$W = a_2\,\delta_2 + b_2\,\delta_2^3 \qquad (2)$$
$$W = k_{eq}\,\delta_{st} \qquad (3)$$
$$\delta_{st} = \delta_1 + \delta_2 \qquad (4)$$

Solve Eqs. (1) and (2) for δ_1 and δ_2, respectively. Substitute the result in Eq. (4) and then in Eq. (3) to find k_{eq}.



Springs in parallel :

$$W = F_1 + F_2$$
$$= a_1\,\delta_{st} + b_1\,\delta_{st}^3 + a_2\,\delta_{st} + b_2\,\delta_{st}^3$$
$$= k_{eq}\,\delta_{st}$$



$$k_{eq} = a_1 + b_1\,\delta_{st}^2 + a_2 + b_2\,\delta_{st}^2$$

(1.21) $k = \frac{G d^4}{8 D^3 N} \geq 8\times10^6$ N/m ; $\frac{D}{d} \geq 6$; $N \geq 10$

$W = \pi D N \rho \left(\frac{\pi d^2}{4}\right)$ where ρ = weight per unit volume

$f_1 = \frac{1}{2}\sqrt{\frac{kg}{W}} = \frac{1}{2}\sqrt{\frac{G d^2 g}{2\pi^2 D^4 N^2 \rho}} \geq 0.4$ Hz

Using $G = 73.1\times10^9$ N/m², $\rho = 76000$ N/m³, $g = 9.81$ m/sec², $\frac{D}{d} = 6, 8, 10$; $N = 10, 15, 20$; $d = 0.4, 0.6, \ldots,$ values of k and f_1 are computed.

Combination of $\frac{D}{d} = 6$, $N = 10$ and $d = 2.0$ m, corresponding to $k = 8.4606\times10^6$ N/m and $f_1 = 0.4801$ Hz, can be taken as an acceptable design.

(1.22) Total elongation (strain) is same in each material:

$$\epsilon_s = \epsilon_a = \frac{x}{\ell} \tag{1}$$

where x is the total elongation. Equation (1) can be expressed as

$$\frac{\sigma_s}{E_s} = \frac{\sigma_a}{E_a} = \frac{x}{\ell} \tag{2}$$

$$\text{or} \quad \sigma_s = \frac{E_s\, x}{\ell} \tag{3}$$

$$\sigma_a = \frac{E_a\, x}{\ell} \tag{4}$$

Total axial force is:

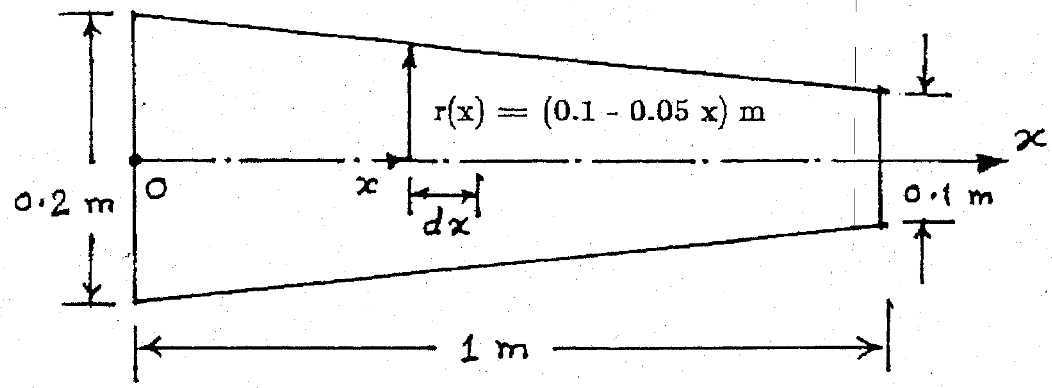
$$F = F_s + F_a = \sigma_s\, A_s + \sigma_a\, A_a \tag{5}$$

where F_s and F_a denote the axial forces acting on steel and aluminum, respectively, and A_s and A_a represent the cross-sectional areas of the two materials. Equating F to $k_{eq}\, x$ where k_{eq} denotes the equivalent spring constant of the bimetallic bar, we obtain from Eqs. (3) to (5):

$$F = k_{eq}\, x = \left(\frac{E_s\, x}{\ell}\right) A_s + \left(\frac{E_a\, x}{\ell}\right) A_a$$

$$\text{or} \quad k_{eq} = \frac{E_s\, A_s}{\ell} + \frac{E_a\, A_a}{\ell} \tag{6}$$

1.23



$J = \dfrac{\pi}{2} r^4 =$ area polar moment of inertia at section $x = 1.5708\ (0.1 - 0.05\ x)^4\ m^4$

Knowing that the angle of twist, θ, between the ends of a uniform shaft of length ℓ under a torque T is given by $\theta = \dfrac{T\ \ell}{GJ}$, the angle of twist for an element of length dx can be expressed as

$$d\theta = \frac{T\ dx}{GJ} = \frac{T\ dx}{(80\ (10^9))\ 1.5708\ (0.1 - 0.05\ x)^4} \tag{1}$$

The total angle of twist can be determined by integrating Eq. (1) from x=0 to 1 as:

$$\theta = \int_0^1 \frac{T\ dx}{(12.5664\ (10^{10}))\ (0.1 - 0.05\ x)^4} = \left(\frac{T}{12.5664\ (10^{10})} \right) \int_0^1 \frac{dx}{(0.1 - 0.05\ x)^4} \tag{2}$$

$$\text{But} \int_0^1 \frac{dx}{(0.1 - 0.05\ x)^4} = - \frac{1}{0.05} \int_0^1 \frac{(-\ 0.05\ dx)}{(0.1 - 0.05\ x)^4} = -\ 20 \int_0^{-0.05} \frac{dy}{(0.1 + y)^4}$$

$= 4.6667\ (10^4)$ where $y = -\ 0.05\ x$

$$\text{Hence}\ \ \theta = \frac{T\ (4.6667)\ (10^4)}{12.5664\ (10^{10})} = T\ (0.3714\ (10^{-6}))\ \ \text{rad}$$

This gives $k_t = \dfrac{T}{\theta} = 2.6925\ (10^6)$ N-m/rad

1.24 The steel and aluminum hollow shafts can be treated as two torsional springs in parallel. For a hollow shaft,

$k_{eq} = \dfrac{\pi\ G}{32\ \ell}\ (D^4 - d^4)$

For the steel shaft, $G = 80\ (10^9)$ Pa, $\ell = 5$ m, $D = 0.25$ m, $d = 0.15$ m, and hence

$k_{t_1} = \dfrac{\pi\ (8\ (10^{10}))}{32\ (5)}\ (0.25^4 - 0.15^4) = 5.34072\ (10^6)$ N–m/rad

For the aluminum shaft, $G = 26\ (10^9)$ Pa, $\ell = 5$ m, $D = 0.15$ m, $d = 0.1$ m, and hence

$k_{t_2} = \dfrac{\pi\ (26\ (10^9))}{32\ (5)}\ (0.15^4 - 0.10^4) = 0.207395\ (10^6)$ N–m/rad

$k_{eq} = k_{t_1} + k_{t_2} = 5.34072\ (10^6) + 0.20739\ (10^6) = 5.54811\ (10^6)$ N–m/rad

1.25 For helical spring: $k = \dfrac{G\,d^4}{64\,n\,R^3}$

Spring 1: $k_1 = \dfrac{(12 \times 10^6)(2^4)}{64\,(10)(6^3)} = 1{,}388.89$ lb/in

Spring 2: $k_2 = \dfrac{(4 \times 10^6)(1^4)}{64\,(10)(5^3)} = 50.00$ lb/in

(a) Spring 2 inside spring 1 (parallel): $k_{eq} = k_1 + k_2 = 1{,}438.89$ lb/in
(b) Spring 2 on top of spring 1 (series):

$$\frac{1}{k_{eq}} = \frac{1}{k_1} = \frac{1}{k_2} = \frac{k_2 + k_1}{k_1\,k_2}$$

which gives $k_{eq} = 48.2625$ lb/in.

1.26 Assume small angles θ_1 and θ_2; $\quad \theta_2 = \left(\dfrac{p_1}{p_2}\right)\theta_1$

x_1 = horizontal displacement of C.G. of mass $m_1 = \theta_1 r_1$
x_2 = vertical displacement of C.G. of mass $m_2 = \theta_2 r_2 = p_1 \theta_1 r_2 / p_2$
y_1 = horizontal displacement of springs k_1 and $k_2 = \theta_1 (r_1 + l_1)$
y_2 = vertical displacement of springs k_3 and $k_4 = \theta_2 l_2 = p_1 l_2 \theta_1 / p_2$

Equivalence of kinetic energies gives

$$\tfrac{1}{2} J_{eq} (\dot\theta_1)^2 = \tfrac{1}{2} J_1 (\dot\theta_1)^2 + \tfrac{1}{2} J_2 (\dot\theta_2)^2 + \tfrac{1}{2} m_1 (\dot x_1)^2 + \tfrac{1}{2} m_2 (\dot x_2)^2$$

$$\therefore\ J_{eq} = J_1 + J_2 \left(p_1/p_2\right)^2 + m_1 r_1^2 + m_2 r_2^2 \left(p_1/p_2\right)^2$$

Equivalence of potential energies gives

$$\tfrac{1}{2} k_{eq} \theta_1^2 = \tfrac{1}{2} k_{12} y_1^2 + \tfrac{1}{2} k_{34} y_2^2 + \tfrac{1}{2} k_{t1} \theta_1^2 + \tfrac{1}{2} k_{t2} \theta_2^2$$

with $k_{12} = k_1 + k_2$, $\quad k_{34} = k_3 k_4 / (k_3 + k_4)$

$y_1 = \theta_1 (r_1 + l_1)$, $\ y_2 = p_1 l_2 \theta_1 / p_2$ and $\theta_2 = p_1 \theta_1 / p_2$.

$$\therefore\ k_{eq} = (k_1 + k_2)(r_1 + l_1)^2 + \left(\frac{k_3 k_4}{k_3 + k_4}\right)\frac{p_1^2 l_2^2}{p_2^2} + k_{t1} + k_{t2}\frac{p_1^2}{p_2^2}.$$

1.27 $\theta = \dfrac{x}{b}$, $\quad x_1 = \dfrac{x\,a}{b}$.

From equivalence of kinetic energies,

$$\tfrac{1}{2} m_{eq} \dot x^2 = \tfrac{1}{2} m_1 \dot x_1^2 + \tfrac{1}{2} m_2 \dot x^2 + \tfrac{1}{2} J_0 \dot\theta^2$$

$$m_{eq} = m_1 \left(\frac{a}{b}\right)^2 + m_2 + J_0 \left(\frac{1}{b}\right)^2$$

(1.28) Let $\dot{\theta}_i$ = angular velocity of the motor (input)

Angular velocities of different gear sets are:

J_{motor}, J_1	J_2, J_3	J_4, J_5	$\cdots$	J_{2N}, J_{load}
$\dot{\theta}_i$	$\dot{\theta}_i\left(\frac{n_1}{n_2}\right)$	$\dot{\theta}_i\left(\frac{n_1}{n_2}\frac{n_3}{n_4}\right)$		$\dot{\theta}_i\left(\frac{n_1}{n_2}\frac{n_3}{n_4}\cdots\frac{n_{2N-1}}{n_{2N}}\right)$

Equivalence of kinetic energies gives

$$\frac{1}{2} J_{eq}\,\dot{\theta}_i^2 = \frac{1}{2} J_{motor}\,\dot{\theta}_i^2 + \frac{1}{2}\sum_{k=1}^{2N} J_k\,\dot{\theta}_k^2 + \frac{1}{2} J_{load}\,\dot{\theta}_{load}^2$$

$$\therefore J_{eq} = (J_{motor} + J_1) + (J_2 + J_3)\left(\frac{n_1}{n_2}\right)^2 + (J_4 + J_5)\left(\frac{n_1}{n_2}\frac{n_3}{n_4}\right)^2 + \cdots + (J_{2N} + J_{load})\left(\frac{n_1}{n_2}\frac{n_3}{n_4}\cdots\frac{n_{2N-1}}{n_{2N}}\right)^2$$

(1.29) Equivalence of kinetic energies gives

$$\frac{1}{2} J_{eq}\,\dot{\theta}_1^2 = \frac{1}{2} J_1\,\dot{\theta}_1^2 + \frac{1}{2} J_2\,\dot{\theta}_2^2 \qquad \text{where} \qquad \dot{\theta}_2 = \dot{\theta}_1\left(\frac{n_1}{n_2}\right)$$

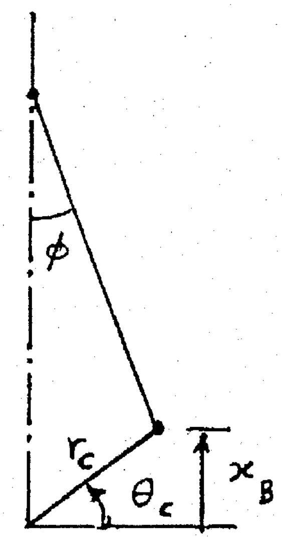
$$J_{eq} = J_1 + J_2\left(\frac{n_1}{n_2}\right)^2$$

(1.30) When point A moves by distance $x = x_h$, the walking beam rotates by the angle $\theta_b = \dfrac{x_h}{\ell_3}$.

This corresponds to a linear motion of point B: $x_B = \theta_b\,\ell_2 = \dfrac{x_h\,\ell_2}{\ell_3}$

and the angular rotation of crank can be found from the relation:

$$x_B = r_c \sin\theta_c + \ell_4 \cos\phi = r_c \sin\theta_c + \ell_4 \sqrt{1 - \frac{r_c^2}{\ell_4^2}\sin^2\theta_c}$$



For large values of ℓ_4 compared to r_c and for small values of x and θ_c, we have

$$x_B \approx r_c \sin\theta_c = r_c\,\theta_c \text{ or } \theta_c = \frac{x_B}{r_c} = \frac{x_h\,\ell_2}{\ell_3\,r_c}$$

The kinetic energy of the system can be expressed as

$$T = \frac{1}{2} m_h\,\dot{x}_h^2 + \frac{1}{2} J_b\,\dot{\theta}_b^2 + \frac{1}{2} J_c\,\dot{\theta}_c^2$$

Equating this to $T = \dfrac{1}{2} m_{eq}\,\dot{x}^2 = \dfrac{1}{2} m_{eq}\,\dot{x}_h^2$, we obtain

$$m_{eq} = m_h + \frac{J_b}{\ell_3^2} + J_c\left(\frac{\ell_2}{\ell_3\,r_c}\right)^2.$$

(1.31) When mass m is displaced by x, the bell crank lever rotates by the angle $\theta_b = \dfrac{x}{\ell_1}$. This makes the center of the sphere displace by $x_s = \theta_b\,\ell_2$. Since the sphere rotates with out slip, it rotates by an angle

$$\theta_s = \frac{x_s}{r_s} = \frac{\theta_b\,\ell_2}{r_s} = \frac{x\,\ell_2}{\ell_1\,r_s}$$

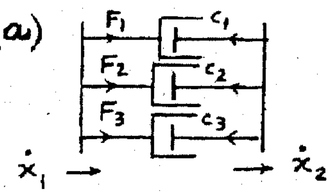
The kinetic energy of the system can be expressed as

$$T = \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,J_0\,\dot{\theta}^2 + \frac{1}{2}\,J_s\,\dot{\theta}_s^2 = \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,J_0\left(\frac{\dot{x}}{\ell_1}\right)^2 + \frac{1}{2}\left(\frac{2}{5}\,m_s\,r_s^2\right)\dot{x}^2\left(\frac{\ell_2}{\ell_1\,r_s}\right)^2$$

since for a sphere, $J_s = \frac{2}{5}\,m_s\,r_s^2$. Equating this to $T = \frac{1}{2}\,m_{eq}\,\dot{x}^2$, we obtain

$$m_{eq} = m + J_0\,\frac{1}{\ell_1^2} + \frac{2}{5}\,m_s\,\frac{\ell_2^2}{\ell_1^2}$$

1.32 (a)



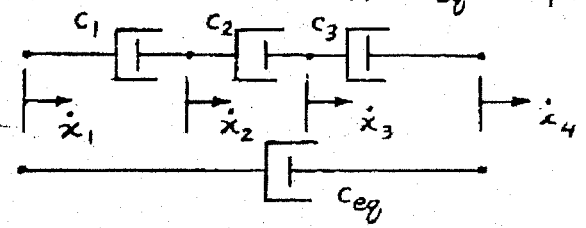
F_i = damping force of $c_i = c_i\,(\dot{x}_2 - \dot{x}_1)$; $i = 1,2,3$

F_{eq} = damping force of $c_{eq} = c_{eq}\,(\dot{x}_2 - \dot{x}_1)$

$\equiv F_1 + F_2 + F_3$

$\therefore\ c_{eq} = c_1 + c_2 + c_3$

(b)



$F_1 = c_1\,(\dot{x}_2 - \dot{x}_1)$

$F_2 = c_2\,(\dot{x}_3 - \dot{x}_2)$

$F_3 = c_3\,(\dot{x}_4 - \dot{x}_3)$

$$\dot{x}_4 - \dot{x}_1 = \dot{x}_4 - \dot{x}_3 + \dot{x}_3 - \dot{x}_2 + \dot{x}_2 - \dot{x}_1$$

$$\frac{F_{eq}}{c_{eq}} = \frac{F_3}{c_3} + \frac{F_2}{c_2} + \frac{F_1}{c_1}$$

since $F_{eq} = F_1 = F_2 = F_3$, $\quad \frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}$

(c) Equating the energies dissipated in a cycle,

$$\pi\,c_{eq}\,\omega\,X_1^2 = \pi\,c_1\,\omega\,X_1^2 + \pi\,c_2\,\omega\,X_2^2 + \pi\,c_3\,\omega\,X_3^2$$

where $X_1 = \theta\,\ell_1$, $X_2 = \theta\,\ell_2$ and $X_3 = \theta\,\ell_3$

$$\therefore\ c_{eq} = c_1 + c_2\left(\frac{\ell_2}{\ell_1}\right)^2 + c_3\left(\frac{\ell_3}{\ell_1}\right)^2$$

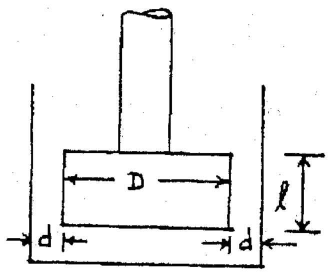
(d) Equating the energies dissipated in a cycle,

$$\pi\,c_{t_{eq}}\,\omega\,\theta_1^2 = \pi\,c_{t1}\,\omega\,\theta_1^2 + \pi\,c_{t2}\,\omega\,\theta_2^2 + \pi\,c_{t3}\,\omega\,\theta_3^2$$

where $\theta_2 = \theta_1\left(\frac{n_1}{n_2}\right)$ and $\theta_3 = \theta_1\left(\frac{n_1}{n_3}\right)$.

$$\therefore\ c_{t_{eq}} = c_{t1} + c_{t2}\left(\frac{n_1}{n_2}\right)^2 + c_{t3}\left(\frac{n_1}{n_3}\right)^2.$$

1.33 Damping constant desired $= c = 1$ lb-sec/in, viscosity of the fluid $= \mu = 4\ \mu$ reyn $= 4\,(10^{-6})$ lb–sec/in^2.



$$c = \mu \left\{ \frac{3\,\pi\, D^3\, \ell\, (1 + \frac{2\,d}{D})}{4\,d^3} \right\} \tag{1}$$

Assuming $x = D/d$ as the unknown with $\ell = 2$ in, Eq. (1) can be written as

$$c = \mu \left(\frac{3\,\pi\,\ell\, x^3}{4} \right) (1 + \frac{2}{x}) \quad \text{or} \quad 1 = (4\,(10^{-8}))\,(\frac{3\,\pi\,(2)}{4})\, x^3\, (1 + \frac{2}{x}) \tag{2}$$

This gives $x^3 + 2\,x^2 - 53{,}051.52 = 0$

Using a trial and error procedure, the solution of this cubic equation can be found as $x \approx 36.92$. Using $D = 3$ in, we get $d = 3/36.92 = 0.08126$ in.

1.34 $c = \mu \left\{ \frac{3\pi D^3 \ell}{4\,d^3} \left(1 + 2\,\frac{d}{D}\right) \right\}$;

D = diameter of piston

ℓ = axial length of piston

d = radial clearance

$\mu = 45$ μ reynolds

(from Shigley's Mechanical Engineering Design)

Let $d = 0.001''$, $D = 2.4''$ and above equation gives

$$10^5 = (45 \times 10^{-6}) \left\{ \frac{3\pi\,(2.4)^3\,\ell}{4\,(0.001)^3} \left(1 + \frac{2 \times 0.001}{2.4}\right) \right\}$$

$\therefore\ \ell = 0.6817''$

1.35 Tangential velocity of inner cylinder $= \frac{D}{2}\omega$

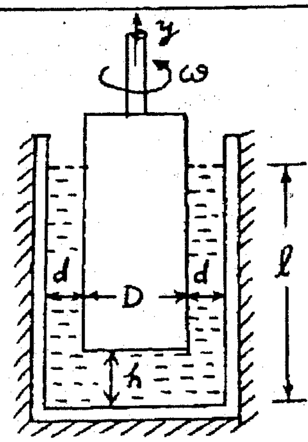
For small d, rate of change of velocity of fluid is $\frac{dv}{dr} = \frac{\frac{D}{2}\omega}{d}$

Shear stress between cylinders is

$$\tau = \mu\,\frac{dv}{dr} = \mu\,\frac{D\omega}{2d}$$

and shear force is

$$F = \tau \cdot \text{Area} = \tau\,\pi D(\ell - h) = \frac{\pi \mu D^2 \omega\,(\ell - h)}{2\,d}$$



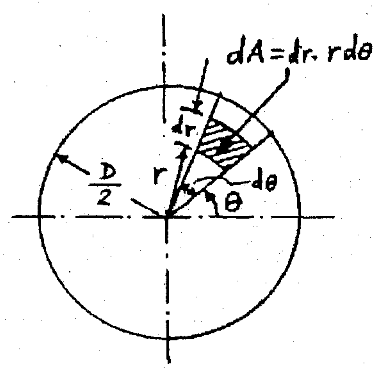
Torque developed $= M_{t1} = F \cdot \frac{D}{2}$

For small h, rate of change of velocity of fluid in vertical direction is

$$\frac{dv}{dy} = \frac{r\,\omega}{h}$$

Shear stress is $\tau = \mu\,\frac{dv}{dy} = \frac{\mu r \omega}{h}$

Force on area $dA = dF = \tau\, dA$



Torque between bottom surfaces of cylinders is

$$M_{t2} = \iint_{area} dM_{t2} \cdot dA \quad \text{where} \quad dM_{t2} = dF \cdot r = \frac{\mu r^3 \omega}{h}\, dr\, d\theta$$

i.e., $$M_{t2} = \frac{\mu \omega}{h} \int_{r=0}^{D/2} \int_{\theta=0}^{2\pi} r^3 . dr\, d\theta = \frac{\mu \omega \pi D^4}{64 h}$$

$$\text{Total torque} = M_t = M_{t1} + M_{t2} = \frac{\pi \mu D^3 \omega (l-h)}{4 d} + \frac{\pi \mu \omega D^4}{64 h}$$

Expressing M_t as $c_t v = c_t \omega D/2$, we get damping constant:

$$c_t = \frac{\pi \mu D^2 (l-h)}{2 d} + \frac{\pi \mu D^3}{32 h}$$

1.36 $F = a \dot{x} + b \dot{x}^2 = 5 \dot{x} + 0.2 \dot{x}^2$

$$F(\dot{x}) \approx F(\dot{x}_0) + \frac{dF}{d\dot{x}} \Big|_{\dot{x}_0} (\dot{x} - \dot{x}_0)$$

At $\dot{x}_0 = 5$ m/s, $F(\dot{x}_0) = 5\ (5) + 0.2\ (25) = 30$ N, $\dfrac{dF}{d\dot{x}} \Big|_{\dot{x}_0} = (5 + 0.4\ \dot{x}) \Big|_5 = 7$ and hence

$F(\dot{x}) = 30 + 7\ (\dot{x} - 5) = 7\ \dot{x} - 5.$

Thus the linearized damping constant is given by $F(\dot{x}) \approx 7\ \dot{x} = c_{eq}\ \dot{x}$ or $c_{eq} = 7$ N–s/m.

1.37 Damping constant due to skin friction drag is:

$$c = 100\ \mu\ \ell^2\ d \qquad (1)$$

Damping constant of a plate-type damper is:

$$c_p = \frac{\mu A}{h} \qquad (2)$$

where A = area of plates and h = distance between the plates. If the area of plates (A) in Fig. 1.35 is taken to be same as the area of the plate shown in Fig. 1.82, we have A = ℓ d. Equating (1) and (2) gives

$$100\ \mu\ \ell^2\ d = \frac{\mu\ \ell\ d}{h} \qquad (3)$$

from which the clearance between the plates can be determined as $h = \dfrac{1}{100\ \ell}$.

1.38

$$c = \frac{6\ \pi\ \mu\ \ell}{h^3} \left\{ (a - \frac{h}{2})^2 - r^2 \right\} \left(\frac{a^2 - r^2}{a - \dfrac{h}{2}} - h \right)$$

When $\mu = 0.3445$ Pa-s, $\ell = 0.1$ m, h = 0.001 m, a = 0..02 m, and r = 0.005 m:

$$c = \frac{6\ \pi\ (0.3445)\ (0.1)}{(10^{-3})^3} \left\{ (0.02 - 0.0005)^2 - 0.005^2 \right\} \left(\frac{0.02^2 - 0.005^2}{0.02 - 0.0005} - 0.001 \right)$$
$$= 4{,}205.6394 \text{ N–s/m}$$

(1.39) $\vec{x} = 5 + 2i = A e^{i\theta} = A \cos\theta + i A \sin\theta$

$A \cos\theta = 5$

$A \sin\theta = 2$

$$A = \sqrt{(A\cos\theta)^2 + (A\sin\theta)^2} = \sqrt{5^2 + 2^2} = 5.3852$$

$$\theta = \tan^{-1}\left(\frac{A\sin\theta}{A\cos\theta}\right) = \tan^{-1}\left(\frac{2}{5}\right) = 21.8014^\circ$$

(1.40) $\vec{x}_1 = 1 + 2i = a_1 + a_2 i \quad , \quad \vec{x}_2 = 3 - 4i = b_1 + b_2 i$

$\vec{x} = \vec{x}_1 + \vec{x}_2 = (a_1 + b_1) + i(a_2 + b_2) = 4 - 2i$

$= A e^{i\theta} = A\cos\theta + i A \sin\theta$

$A = \sqrt{4^2 + (-2)^2} = 4.4721$

$\theta = \tan^{-1}\left(\frac{-2}{4}\right) = -26.5651^\circ$

(1.41) $z_1 = (3 - 4i),\ z_2 = (1 + 2i)$

$z = z_1 - z_2 = (3 - 4i) - (1 + 2i) = 2 - 6i = A e^{i\theta}$

where $A = \sqrt{2^2 + (-6)^2} = 6.3246$ and $\theta = \tan^{-1}\left(\frac{-6}{2}\right) = \tan^{-1}(-3) = -1.2490$ rad

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(1.42) $z_1 = 1 + 2i,\quad z_2 = 3 - 4i$

$z = z_1 z_2 = (1 + 2i)(3 - 4i) = 11 + 2i = A e^{i\theta}$

where $A = \sqrt{11^2 + 2^2} = 11.1803$ and $\theta = \tan^{-1}(2/11) = 0.1798$ rad

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(1.43) $z = \frac{z_1}{z_2} = \frac{1 + 2i}{3 - 4i} = \frac{(1 + 2i)(3 + 4i)}{(3 - 4i)(3 + 4i)} = \frac{-5 + 10i}{25} = -0.2 + 0.4i = A e^{i\theta}$

where $A = \sqrt{(-0.2)^2 + (0.4)^2} = 0.4472$

and $\theta = \tan^{-1}\left(\frac{-0.4}{0.2}\right) = \tan^{-1}(-2) = -1.1071$ rad

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(1.44) $x(t) = X\cos\omega t\ ,\quad y(t) = Y\cos(\omega t + \phi)$

(a) $\frac{x^2}{X^2} = \cos^2\omega t,\ \frac{y^2}{Y^2} = \cos^2(\omega t + \phi),$

$2\frac{xy}{XY}\cos\phi = 2\cos\omega t\cos(\omega t + \phi)\cos\phi$

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\frac{xy}{XY}\cos\phi$$
$$= \cos^2\omega t + \cos^2(\omega t + \phi) - 2\cos\omega t\cos\phi\cos(\omega t + \phi) \tag{1}$$

Noting that $\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha)$, Eq. (1) can be rewritten as

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\frac{xy}{XY}\cos\phi$$
$$= \frac{1}{2} + \frac{1}{2}\cos 2\omega t + \frac{1}{2} + \frac{1}{2}\cos(2\omega t + 2\phi) - 2\cos\omega t\cos\phi\cos(\omega t + \phi)$$

$$= 1 + \frac{1}{2}\left\{2\cos\frac{2\,\omega\,t + 2\,\omega\,t + 2\,\phi}{2}\cos\frac{2\,\omega\,t - 2\,\omega\,t - 2\,\phi}{2}\right\}$$
$$- 2\cos\omega\,t\cos\phi\cos(\omega\,t + \phi)$$
$$= 1 + \cos(2\,\omega\,t + \phi)\cos\phi - 2\cos\omega\,t\cos\phi\cos(\omega\,t + \phi)$$
$$= 1 + \cos(2\,\omega\,t + \phi)\cos\phi - 2\cos\phi\left\{\frac{1}{2}\left[\cos(\omega\,t + \phi - \omega\,t) + \cos(\omega\,t + \phi + \omega\,t)\right]\right\}$$
$$= 1 + \cos\phi\cos(2\,\omega\,t + \phi) - \cos\phi\left\{\cos\phi + \cos(2\,\omega\,t + \phi)\right\}$$
$$= 1 - \cos^2\phi = \sin^2\phi \qquad (2)$$

(b) When $\phi = 0$, Eq. (2) reduces to

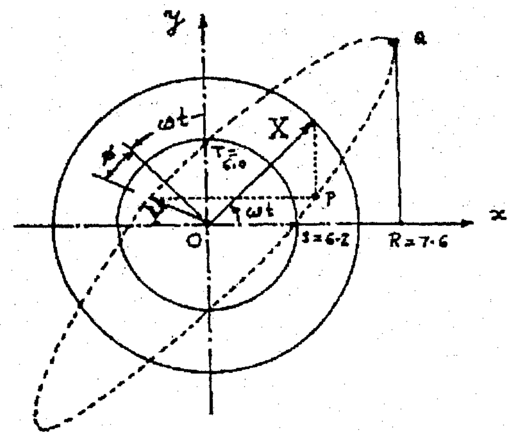
$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\,\frac{x\,y}{X\,Y} = \left(\frac{x}{X} - \frac{y}{Y}\right)^2 = 0$$

which gives $X = \pm\frac{X}{Y}\,y$. This indicates that the locus of the resultant motion is a straight line. When $\phi = \frac{\pi}{2}$, Eq. (2) reduces to

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} = 1$$

which denotes an ellipse with its major and minor axes along x and y directions, respectively. When $\phi = \pi$, Eq. (2) reduces to that of a straight line as in the case of $\phi = 0$.

1.45



Equation for resultant motion:

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\,\frac{x\,y}{X\,Y}\cos^2\phi = \sin^2\phi \qquad (1)$$

When $y = 0$, Eq. (1) reduces to $\frac{x^2}{X^2} = \sin^2\phi$ and hence:

$$x = \pm X\sin\phi = \pm 6.2 = \text{OS in figure} \qquad (2)$$

When $x = 0$, Eq. (1) reduces to $\frac{y^2}{Y^2} = \sin^2\phi$ and hence:

$$y = \pm Y\sin\phi = \pm 6.0 = \text{OT in figure} \qquad (3)$$

It can be seen that

$$\text{OR} = X\cos\phi = 7.6 \text{ in figure} \qquad (4)$$

$$\frac{\text{OS}}{\text{OR}} = \frac{X\sin\phi}{X\cos\phi} = \tan\phi = \frac{6.2}{7.6} = 0.8158 \quad \text{or} \quad \phi = 39.2072^\circ \qquad (5)$$

From Eqs. (2) and (4), we find

$$X = \sqrt{(X\sin\phi)^2 + (X\cos\phi)^2} = \sqrt{(6.2)^2 + (7.6)^2} = 9.8082 \text{ mm}$$

Equations (3) and (5) give

$$Y = \frac{6.0}{\sin \phi} = \frac{6.0}{\sin 39.2072^\circ} = 9.4918 \text{ mm}$$

1.46 (a) $x(t) = \frac{A}{1000} \cos(50t + \alpha)$ m where A is in mm ---- (E_1)

$x(0) = \frac{A}{1000} \cos \alpha = 0.003$, $A \cos \alpha = 3$ ---- (E_2)

$\dot{x}(0) = -\frac{50A}{1000} \sin \alpha = 1$, $A \sin \alpha = -20$ ---- (E_3)

$$A = \left\{ (A \cos \alpha)^2 + (A \sin \alpha)^2 \right\}^{1/2} = 20.2237 \text{ mm}$$

$$\alpha = \tan^{-1} \left(\frac{A \sin \alpha}{A \cos \alpha} \right) = \tan^{-1}(-6.6667) = -81.4692^\circ = -1.4219 \text{ rad}$$

$$x(t) = 20.2237 \cos(50t - 1.4219) \text{ mm}$$

(b) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Eq. (E_1) can be expressed as $x(t) = A \cos 50t \cdot \cos \alpha - A \sin 50t \cdot \sin \alpha$

$= A_1 \cos \omega t + A_2 \sin \omega t$

where $\omega = 50$, $A_1 = A \cos \alpha$, $A_2 = -A \sin \alpha$

$\therefore x(t) = (3 \cos 50t + 20 \sin 50t)$ mm

1.47 $x(t) = A_1 \cos \omega t + A_2 \sin \omega t$

$\frac{dx}{dt}(t) = -A_1 \omega \sin \omega t + A_2 \omega \cos \omega t$, $\frac{d^2x}{dt^2} = -A_1 \omega^2 \cos \omega t - A_2 \omega^2 \sin \omega t$

$\frac{d^2x}{dt^2} = -\omega^2 x(t)$ where ω^2 is a constant

Hence $x(t)$ is a simple harmonic motion.

1.48 (a) Using trigonometric relations:

$x_1(t) = 5 (\cos 3t \cos 1 - \sin 3t \sin 1)$

$x_2(t) = 10 (\cos 3t \cos 2 - \sin 3t \sin 2)$

$x(t) = x_1(t) + x_2(t) = \cos 3t (5 \cos 1 + 10 \cos 2) - \sin 3t (5 \sin 1 + 10 \sin 2)$

If $x(t) = A \cos(\omega t + \alpha) = A \cos \omega t \cos \alpha - A \sin \omega t \sin \alpha$,

$\omega = 3$, $A \cos \alpha = 5 \cos 1 + 10 \cos 2 = -1.4599$,

$A \sin \alpha = 5 \sin 1 + 10 \sin 2 = 13.3003$

$$A = \sqrt{(A \cos \alpha)^2 + (A \sin \alpha)^2} = 13.3802$$

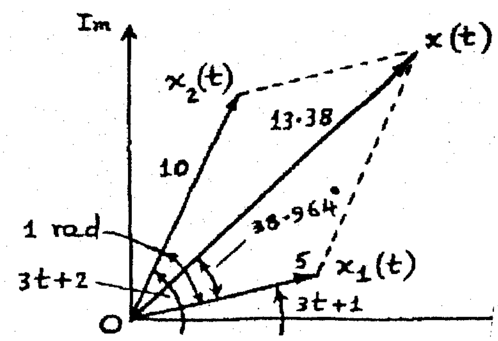
$$\alpha = \tan^{-1} \left(\frac{A \sin \alpha}{A \cos \alpha} \right) = \tan^{-1}(-9.1104) = 96.2640^\circ = 1.68 \text{ rad}$$

Angle between $x_1(t)$ and $x(t)$ is $96.2640^\circ - 57.3^\circ = 38.964^\circ$

(b) Using vector addition:

For an arbitrary value of $(\omega t + 1)$, harmonic motions $x_1(t)$ and $x_2(t)$ can be shown as in the figure. From vector addition, we find

$\simeq 13.38 \cos(\omega t + 1.68)$



(c) Using complex numbers:

$$x_1(t) = \text{Re}\left\{A_1\, e^{i(\omega t+1)}\right\} = \text{Re}\left\{5\, e^{i(\omega t+1)}\right\}$$

$$x_2(t) = \text{Re}\left\{A_2\, e^{i(\omega t+2)}\right\} = \text{Re}\left\{10\, e^{i(\omega t+2)}\right\}$$

If $x(t) = \text{Re}\left\{A\, e^{i(\omega t+\alpha)}\right\}$,

$$A\cos(3t+\alpha) = A_1\cos(3t+1) + A_2\cos(3t+2)$$

i.e. $$A(\cos 3t\cos\alpha - \sin 3t\sin\alpha) = 5(\cos 3t\cdot\cos 1 - \sin 3t\cdot\sin 1) + 10(\cos 3t\cdot\cos 2 - \sin 3t\cdot\sin 2)$$

i.e. $A\cos\alpha = 5\cos 1 + 10\cos 2$, $\quad A\sin\alpha = 5\sin 1 + 10\sin 2$

$A = 13\cdot3802$, $\alpha = 1\cdot68$ rad

$$x(t) = \text{Re}\left\{13\cdot3802\, e^{i(3t+1\cdot68)}\right\}$$

(1.49) $x(t) = 10\sin(\omega t+60^\circ) = x_1(t) + x_2(t)$

where $x_1(t) = 5\sin(\omega t+30^\circ)$ and $x_2(t) = A\sin(\omega t+\alpha^\circ)$

$$10(\sin\omega t\cos 60^\circ + \cos\omega t\sin 60^\circ) = 5(\sin\omega t\cdot\cos 30^\circ + \cos\omega t\cdot\sin 30^\circ) + A(\sin\omega t\cdot\cos\alpha^\circ + \cos\omega t\cdot\sin\alpha^\circ)$$

$10\cos 60^\circ = 5\cos 30^\circ + A\cos\alpha^\circ$; $\quad A\cos\alpha^\circ = 0\cdot6699$

$10\sin 60^\circ = 5\sin 30^\circ + A\sin\alpha^\circ$; $\quad A\sin\alpha^\circ = 6\cdot1603$

$$A = \sqrt{0\cdot6699^2 + 6\cdot1603^2} = 6\cdot1966$$

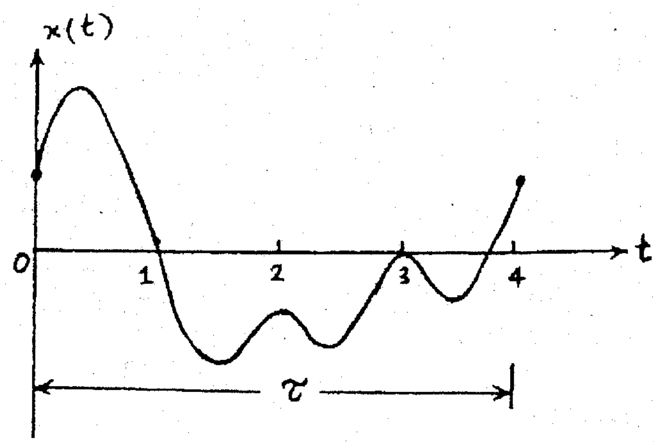
$$\alpha = \tan^{-1}(6\cdot1603/0\cdot6699) = 83\cdot7938^\circ$$

$$x_2(t) = 6\cdot1966\sin(\omega t + 83\cdot7938^\circ)$$

(1.50) $x(t) = \frac{1}{2}\cos\frac{\pi}{2}t + \sin\pi t$

$$= \frac{1}{2}\cos\frac{\pi}{2}t\left(1 + 4\sin\frac{\pi}{2}t\right)$$

From the nature of the graph of $x(t)$, it can be seen that $x(t)$ is periodic with a time period of $\tau = 4$.



(1.51) If $x(t)$ is harmonic, $\ddot{x}(t) = -\omega^2 x(t)$

Here $x(t) = 2\cos 2t + \cos 3t$

$\ddot{x}(t) = -8\cos 2t - 9\cos 3t \neq -$ constant times $x(t)$

$\therefore$ $x(t)$ is not harmonic

(1.52) $x(t) = \frac{1}{2}\cos\frac{\pi}{2}t - \cos\pi t$

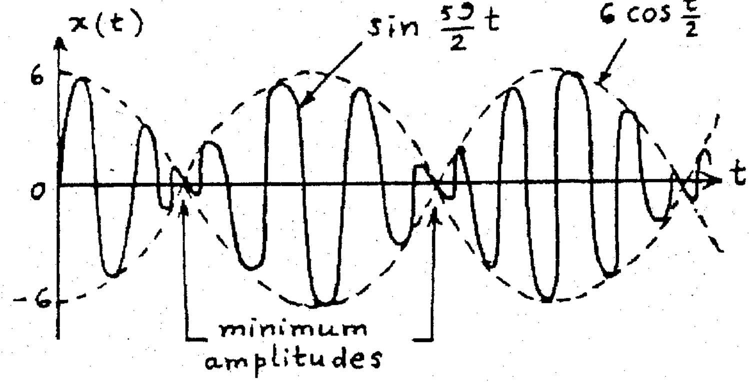
$\ddot{x}(t) = -\frac{\pi^2}{8}\cos\frac{\pi}{2}t + \pi^2\cos\pi t \neq -$ constant times $x(t)$

$\therefore$ $x(t)$ is not harmonic

(1.53) $x(t) = x_1(t) + x_2(t) = 3\sin 30t + 3\sin 29t$

Since $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$,

$x(t) = \left(6\cos\frac{t}{2}\right)\sin\frac{59}{2}t$



This equation shows that the amplitude $(6\cos\frac{t}{2})$ varies with time between a maximum value of 6 and a minimum value of 0. The frequency of this oscillation (beat frequency) is $\omega_b = 1$.

Note: Beat frequency is twice the frequency of the term $6\cos\frac{t}{2}$ since two peaks pass in each cycle of $(6\cos\frac{t}{2})$.

(1.54) The resultant motion of two harmonic motions having identical amplitudes (X) but slightly different frequencies (ω and $\omega + \delta\omega$) is given by Eq. (1.67):

$$x(t) = 2\,X\sin\left(\omega\,t + \frac{\delta\omega\,t}{2}\right)\cos\left(\frac{\delta\,\omega\,t}{2}\right)$$

Thus the maximum amplitude of the resultant motion is equal to 2X and the beat frequency is equal to $\delta\omega$. From Fig. 1.86, we find that $2X \approx 5$ mm or $X = 2.5$ mm and

$$\frac{\delta\omega}{2} = \frac{2\,\pi}{\tau_{\text{beat}}} = \frac{2\,\pi}{\tau_{\text{larger}}} = \frac{2\,\pi}{2\,(12.6 - 4.2)} = 0.374 \text{ rad/sec}$$

or $\delta\omega = 0.748$ rad/sec and $\omega + \frac{\delta\,\omega}{2} = \frac{2\,\pi}{\tau_{\text{smaller}}} = \frac{2\,\pi}{1} = 6.2832$ rad/sec

Hence $\omega = 6.2832 - 0.3740 = 5.9092$ rad/sec. Thus the amplitudes of the two motions $= X = 2.5$ mm and their frequencies are $\omega = 5.9092$ rad/sec and $\omega + \delta\omega = 5.9092 + 0.7480 = 6.6572$ rad/sec.

(1.55) $A = 0.05$ m, $\omega = 10$ Hz $= 62.832$ rad/sec

period $= \tau = \frac{2\pi}{\omega} = \frac{2\pi}{62.832} = 0.1$ sec

maximum velocity $= A\omega = 0.05 \times 62.832 = 3.1416$ m/s

maximum acceleration $= A\omega^2 = 0.05\,(62.832)^2 = 197.393$ m/s^2

(1.56) $\omega = 15$ cps $= 94.248$ rad/sec

$\ddot{x}_{max} = 0.5 g = 0.5\,(9.81) = 4.905$ m/s^2 $= A\,\omega^2$

$A =$ amplitude $= 4.905/(94.248)^2 = 0.0005522$ m

$\dot{x}_{max} =$ max. velocity $= A\,\omega = 0.05204$ m/s

(1.57) $x = A \cos \omega t$, $\quad x_{max} = A = 0.25$ mm , $\quad \ddot{x} = -\omega^2 A \cos \omega t$

$\ddot{x}_{max} = A\omega^2 = 0.4 g = 3924$ mm/s^2 ; $\omega^2 = 3924/A = 15696$ (rad/s)^2

Operating speed of pump $= \omega = 125.2837$ rad/s $= 19.9395$ rpm

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(1.58)
$$x(t) = X \sin \frac{2\pi t}{\tau} \; ; \; x_{rms} = \left[\frac{1}{\tau} \int_0^{\tau} X^2 \,\text{sun}^2 \frac{2\pi t}{\tau}\, dt \right]^{\frac{1}{2}}$$

Using $\sin^2 \dfrac{2\pi t}{\tau} = \dfrac{1 - \cos \dfrac{4\pi t}{\tau}}{2}$, we obtain

$$x_{rms} = \left[\frac{X^2}{\tau} \int_0^{\tau} \left(\frac{1}{2} - \frac{1}{2} \cos \frac{4\pi t}{\tau} \right) dt \right]^{\frac{1}{2}} = \left[\frac{X^2}{\tau} \left\{ \frac{t}{2} - \frac{1}{2}\,\frac{\tau}{4\pi} \sin \frac{4\pi t}{\tau} \right\} \Big|_0^{\tau} \right]^{\frac{1}{2}}$$

$$= \left[\frac{X^2}{\tau} \left\{ \frac{\tau}{2} - \frac{\tau}{8\pi} \sin 4\pi - 0 + 0 \right\} \right]^{\frac{1}{2}} = \frac{X}{\sqrt{2}}$$

(1.59) $x(t) = \dfrac{A\,t}{\tau}$; $0 \le t \le \tau$

$$x_{rms} = \left\{ \frac{1}{\tau} \int_0^{\tau} \frac{A^2}{\tau^2}\, t^2\, dt \right\}^{\frac{1}{2}} = \left\{ \frac{1}{\tau}\, \frac{A^2}{\tau^2} \left(\frac{t^3}{3} \right) \Big|_0^{\tau} \right\}^{\frac{1}{2}} = \left\{ \frac{A^2}{\tau^3}\, \frac{\tau^3}{3} \right\}^{\frac{1}{2}} = \left(\frac{A^2}{3} \right)^{\frac{1}{2}} = \frac{A}{\sqrt{3}}$$

(1.60) For even functions, $x(-t) = x(t)$.

From Eq. (1.73), $\quad b_n = \dfrac{2}{\tau} \displaystyle\int_0^{\tau} x(t) \sin n\omega t \cdot dt = \dfrac{2}{\tau} \int_{-\tau/2}^{\tau/2} x(t) \sin n\omega t \; dt$

$$= \frac{2}{\tau} \left[\int_{-\frac{\tau}{2}}^{0} x(t) \sin n\omega t \cdot dt + \int_0^{\tau/2} x(t) \sin n\omega t \cdot dt \right] \quad \text{---} (E_1)$$

Since $\sin(-n\omega t) = -\sin(n\omega t) =$ odd function of t, the product of $x(t)$ and $\sin n\omega t$ is an odd function.

Further, for an odd function $f(t)$, $f(-t) = -f(t)$, and

$$\int_{-a}^{a} f(t)\,dt = \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt = \int_{0}^{a} f(-t)\,dt + \int_{0}^{a} f(t)\,dt$$

$$= -\int_{0}^{a} f(t)\,dt + \int_{0}^{a} f(t)\,dt = 0 \qquad \text{----}(E_2)$$

Equations (E_1) and (E_2) lead to $b_n = 0$.

Also, since $\cos n\omega t$ is an even function, we get

$$a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} x(t)\cos n\omega t\,dt = \frac{4}{\tau}\int_{0}^{\tau/2} x(t)\cos n\omega t\,dt$$

For odd functions, $x(-t) = -x(t)$.

From Eq. (1.72), $a_n = \frac{2}{\tau}\int_{0}^{\tau} x(t)\cos n\omega t\,dt = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} x(t)\cos n\omega t\,dt$

Since $\cos n\omega t$ is an even function, $\cos(-n\omega t) = \cos(n\omega t)$, the product of $x(t)$ and $\cos n\omega t$ is an odd function.

Hence $a_n = 0$.

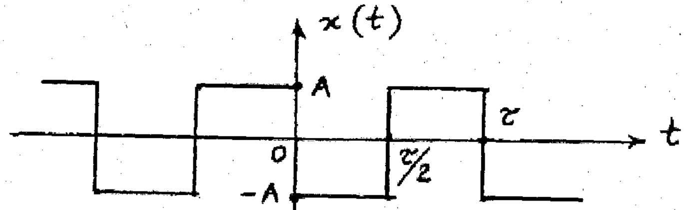
Further, since $\sin n\omega t$ is an odd function, $x(t)\sin n\omega t$ is an even function and hence

$$b_n = \frac{4}{\tau}\int_{0}^{\tau/2} x(t)\sin n\omega t\,dt$$

1.61

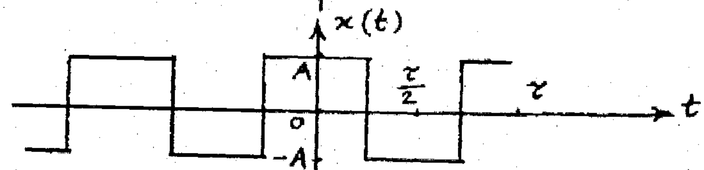
$$x(t) = \begin{cases} -A, & 0 \le t \le \frac{\tau}{2} \\ A, & \frac{\tau}{2} \le t \le \tau \end{cases}$$

(a)



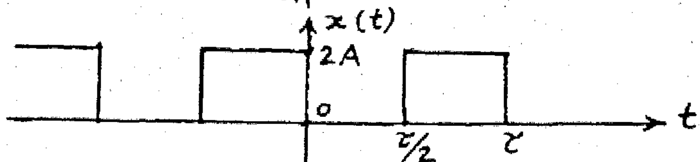
$$x(t) = \begin{cases} A, & 0 \le t \le \tau/4 \\ -A, & \frac{\tau}{4} \le t \le \frac{3\tau}{4} \\ A, & \frac{3\tau}{4} \le t \le \tau \end{cases}$$

(b)



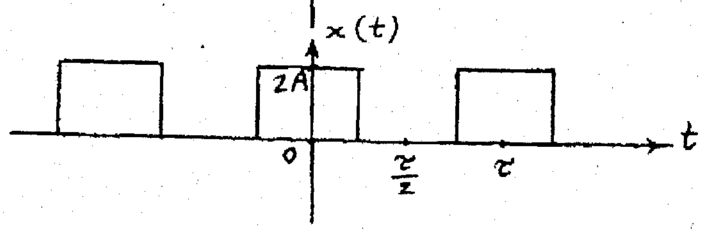
$$x(t) = \begin{cases} 0, & 0 \le t \le \tau/2 \\ 2A, & \frac{\tau}{2} \le t \le \tau \end{cases}$$

(c)



$$x(t) = \begin{cases} 2A, & 0 \le t \le \frac{\tau}{4} \\ 0, & \frac{\tau}{4} \le t \le \frac{3\tau}{4} \\ 2A, & \frac{3\tau}{4} \le t \le \tau \end{cases}$$

(d)



(a) $x(-t) = -x(t)$, odd function, hence $a_0 = a_n = 0$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t \cdot dt = \frac{2}{\tau}\left[-A\int_0^{\tau/2}\sin n\omega t\cdot dt + A\int_{\tau/2}^{\tau}\sin n\omega t\cdot dt\right]$$

$$= -\frac{2A}{\tau}\left(-\frac{\cos n\omega t}{n\omega}\right)_0^{\pi/\omega} + \frac{2A}{\tau}\left(-\frac{\cos n\omega t}{n\omega}\right)_{\pi/\omega}^{2\pi/\omega}$$

$$= \frac{2A}{\tau n\omega}\left(2\cos n\pi - \cos 0 - \cos 2n\pi\right)$$

$$\therefore\ x(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\omega t$$

(b) $x(-t) = x(t)$, even function, hence $b_n = 0$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[A\cdot(t)_0^{\tau/4} - A(t)_{\tau/4}^{3\tau/4} + A(t)_{3\tau/4}^{\tau}\right] = 0$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\cdot dt$$

$$= \frac{2A}{\tau n\omega}\left[\sin n\omega t\Big|_0^{\tau/4} - \sin n\omega t\Big|_{\tau/4}^{\frac{3\tau}{4}} + \sin n\omega t\Big|_{3\tau/4}^{\tau}\right]$$

$$= \frac{A}{n\pi}\left[2\sin\frac{n\pi}{2} - 2\sin\frac{3n\pi}{2} + \sin 2\pi n\right] = \begin{cases} 4A/n\pi & \text{for } n = 1, 5, 9, \dots \\ -4A/n\pi & \text{for } n = 3, 7, 11, \dots \end{cases}$$

$$\therefore\ x(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\cos\frac{2\pi(n-1)t}{\tau}$$

(c) $$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[0 + 2A\ (t)\Big|_{\tau/2}^{\tau}\right] = 2A$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\ dt = \frac{4A}{n\omega\tau}\left(\sin n\omega t\right)_{\tau/2}^{\tau} = 0$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\ dt = -\frac{4A}{n\omega\tau}\left(\cos n\omega t\right)_{\tau/2}^{\tau}$$

$$= -\frac{4A}{n\omega\tau}\left(\cos 2\pi n - \cos n\pi\right)$$

$$\therefore\ x(t) = -\frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\omega t \qquad \text{with } \omega = 2\pi/\tau.$$

(d) $x(-t) = x(t)$, even function, hence $b_n = 0$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[2A\left(\frac{\tau}{4} - 0\right) + 2A\left(\tau - \frac{3\tau}{4}\right)\right] = 2A$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\ dt = \frac{4A}{n\omega\tau}\left[\left(\sin n\omega t\right)_0^{\tau/4} + \left(\sin n\omega t\right)_{\frac{3\tau}{4}}^{\tau}\right]$$

$$= \frac{4A}{n\omega\tau}\left(\sin\frac{n\pi}{2} + \sin 2n\pi - \sin\frac{3n\pi}{2}\right)$$

$$\therefore\ x(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\cos\frac{2\pi(2n-1)t}{\tau} \qquad \text{with } \omega = 2\pi/\tau.$$

(1.62) $x(t) = \begin{cases} A\sin\frac{2\pi t}{\tau}, & 0 \le t \le \frac{\tau}{2} \\ 0, & \frac{\tau}{2} \le t \le \tau \end{cases}$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2A}{\tau}\int_0^{\tau/2}\sin\frac{2\pi t}{\tau}\,dt = \frac{2A}{\tau}\left(-\frac{\tau}{2\pi}\cdot\cos\frac{2\pi t}{\tau}\right)_0^{\tau/2}$$

$$= \frac{2A}{\pi}$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt = \frac{2A}{\tau}\int_0^{\tau/2}\sin\frac{2\pi t}{\tau}\cdot\cos n\omega t\,.\,dt \qquad \text{----}\ (E_1)$$

Using the relation $\sin m\omega t\cdot\cos n\omega t = \frac{\sin(m+n)\omega t + \sin(m-n)\omega t}{2}$, Eq. (E_1) can be rewritten as

$$a_n = \frac{A}{\tau}\int_0^{\pi/\omega}\left[\sin(1+n)\omega t + \sin(1-n)\omega t\right]dt$$

when $n = 1$, $\quad a_1 = \frac{A}{\tau}\int_0^{\pi/\omega}\sin 2\omega t\,.\,dt = 0$

when $n = 2, 3, 4, \ldots$, $\quad a_n = \frac{A}{\tau}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$

$$= \frac{A}{2\pi}\left[\frac{1-\cos(1+n)\pi}{1+n} + \frac{1-\cos(1-n)\pi}{1-n}\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{2A}{(n-1)(n+1)\pi} & \text{if } n \text{ is even} \end{cases}$$

Similarly

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt = \frac{2A}{\tau}\int_0^{\tau/2}\sin\frac{2\pi t}{\tau}\cos n\omega t\,dt$$

$$= \frac{A}{\tau}\int_0^{\tau/2}\left[\cos(1-n)\omega t - \cos(1+n)\omega t\right]dt$$

when $n = 1$, $\quad b_1 = \frac{A}{\tau}\int_0^{\pi/\omega}(dt - \cos 2\omega t)\,dt = \frac{A}{2}$

when $n = 2, 3, 4, \ldots$, $\quad b_n = \frac{A}{\tau}\left[\frac{\sin(1-n)\omega t}{(1-n)\omega} - \frac{\sin(1+n)\omega t}{(1+n)\omega}\right]_0^{\pi/\omega} = 0$

$$\therefore\ x(t) = \frac{A}{\pi} + \frac{A}{2}\sin\omega t - \frac{2A}{\pi}\sum_{n=2,4,6,\ldots}^{\infty}\frac{\cos n\omega t}{(n^2-1)}$$

(1.63) $x(t) = \begin{cases} \frac{2At}{\tau}, & 0 \le t \le \frac{\tau}{2} \\ -\frac{2At}{\tau} + 2A, & \frac{\tau}{2} \le t \le \tau \end{cases}$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[\int_0^{\tau/2} \frac{2At}{\tau}\,dt + \int_{\tau/2}^{\tau}\left(-\frac{2At}{\tau} + 2A\right)dt\right]$$

$$= \frac{2}{\tau}\left[\frac{2A}{\tau}\cdot\frac{t^2}{2}\Big|_0^{\tau/2} - \frac{2A}{\tau}\cdot\frac{t^2}{2}\Big|_{\tau/2}^{\tau} + 2A\cdot t\Big|_{\tau/2}^{\tau}\right]$$

$$= \frac{2}{\tau}\left[\frac{A\tau}{4} - \frac{3A\tau}{4} + A\tau\right] = A$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt$$

$$= \frac{2}{\tau}\left[\int_0^{\frac{\tau}{2}} \frac{2A}{\tau}\,t\cos n\omega t\,dt + \int_{\tau/2}^{\tau}\left(-\frac{2A}{\tau}\,t + 2A\right)\cos n\omega t\,dt\right]$$

$$= \frac{2}{\tau}\left[\frac{2A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_0^{\tau/2}\right.$$

$$\left. - \frac{2A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_{\tau/2}^{\tau} + 2A\left(-\frac{\sin n\omega t}{n\omega}\right)_{\tau/2}^{\tau}\right]$$

As $\tau = \frac{2\pi}{\omega}$,

$$a_n = \frac{\omega}{\pi}\left[\frac{A\omega}{\pi n^2\omega^2}\cos n\pi - \frac{A\omega}{\pi n^2\omega^2} - \frac{A\omega}{\pi n^2\omega^2}\cos 2\pi n + \frac{A\omega}{\pi n^2\omega^2}\cos n\pi\right]$$

$$= \frac{2A}{n^2\pi^2}(\cos n\pi - 1) = \begin{cases} -\frac{4A}{n^2\pi^2}, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots \end{cases}$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt = \frac{2}{\tau}\left[\int_0^{\tau/2}\frac{2A}{\tau}\,t\sin n\omega t\,dt\right.$$

$$\left. + \int_{\tau/2}^{\tau}\left(-\frac{2A}{\tau}\,t + 2A\right)\sin n\omega t\,dt\right]$$

$$= \frac{2}{\tau}\left[\frac{2A}{\tau}\left\{-\frac{t\cos n\omega t}{n\omega} + \frac{\sin n\omega t}{n^2\omega^2}\right\}_0^{\tau/2}\right.$$

$$\left. - \frac{2A}{\tau}\left\{-\frac{t\cos n\omega t}{n\omega} + \frac{\sin n\omega t}{n^2\omega^2}\right\}_{\tau/2}^{\tau} + 2A\left(-\frac{\cos n\omega t}{n\omega}\right)_{\tau/2}^{\tau}\right]$$

$$= \frac{\omega}{\pi}\left[-\frac{A}{n\omega}\cos n\pi + \frac{2A}{n\omega}\cos 2n\pi - \frac{A}{n\omega}\cos n\pi - \frac{2A}{n\omega}\cos 2n\pi + \frac{2A}{n\omega}\cos n\pi\right] = 0$$

$$\therefore\ x(t) = \frac{A}{2} - \frac{4A}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \cos n\omega t$$

(1.64)

$$x(t) = \begin{cases} \dfrac{4At}{\tau} & , \quad 0 \le t \le \dfrac{\tau}{4} \\ -\dfrac{4At}{\tau} + 2A & , \quad \dfrac{\tau}{4} \le t \le \dfrac{3\tau}{4} \\ \dfrac{4At}{\tau} - 4A & , \quad \dfrac{3\tau}{4} \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\, dt = \frac{2}{\tau}\left[\frac{4A}{\tau}\left.\frac{t^2}{2}\right|_0^{\tau/4} + \left(-\frac{4A}{\tau}\frac{t^2}{2} + 2At\right)_{\tau/4}^{3\tau/4} + \left(\frac{4A}{\tau}\frac{t^2}{2} - 4At\right)_{3\tau/4}^{\tau}\right] = 0$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\, dt = \frac{2}{\tau}\left[\frac{4A}{\tau}\int_0^{\tau/4} t\cos n\omega t\cdot dt - \frac{4A}{\tau}\int_{\tau/4}^{3\tau/4} t\cos n\omega t\, dt + 2A\int_{\tau/4}^{3\tau/4}\cos n\omega t\, dt + \frac{4A}{\tau}\int_{3\tau/4}^{\tau} t\cos n\omega t\, dt - 4A\int_{3\tau/4}^{\tau}\cos n\omega t\, dt\right]$$

$$= \frac{2}{\tau}\left[\frac{4A}{\tau}\left\{t\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_0^{\tau/4} - \frac{4A}{\tau}\left\{t\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_{\tau/4}^{3\tau/4} + 2A\left(\frac{\sin n\omega t}{n\omega}\right)_{\tau/4}^{3\tau/4} + \frac{4A}{\tau}\left\{t\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_{3\tau/4}^{\tau} - 4A\left(\frac{\sin n\omega t}{n\omega}\right)_{3\tau/4}^{\tau}\right]$$

$$= \frac{\omega}{\pi}\left[\sin\frac{n\pi}{2}\left(\frac{A}{n\omega} + \frac{A}{n\omega} - \frac{2A}{n\omega}\right) + \cos\frac{n\pi}{2}\left(\frac{2A}{\pi n^2\omega} + \frac{2A}{\pi n^2\omega}\right) + \sin\frac{3n\pi}{2}\left(-\frac{3A}{n\omega} + \frac{2A}{n\omega} - \frac{3A}{n\omega} + \frac{4A}{n\omega}\right) + \cos\frac{3n\pi}{2}\left(-\frac{2A}{\pi n^2\omega} - \frac{2A}{\pi n^2\omega}\right) + \cos 2\pi n\left(\frac{2A}{\pi n^2\omega}\right) - \cos 0\left(\frac{2A}{\pi n^2\omega}\right)\right] = 0$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\, dt = \frac{2}{\tau}\left[\frac{4A}{\tau}\int_0^{\tau/4} t\sin n\omega t\,.dt - \frac{4A}{\tau}\int_{\tau/4}^{3\tau/4} t\sin n\omega t\,.dt + 2A\int_{\tau/4}^{3\tau/4}\sin n\omega t\,.dt + \frac{4A}{\tau}\int_{3\tau/4}^{\tau} t\sin n\omega t\,.dt - 4A\int_{3\tau/4}^{\tau}\sin n\omega t\,.dt\right]$$

$$= \frac{2}{\tau}\left[\frac{4A}{\tau}\left\{\frac{1}{n^2\omega^2}\sin n\omega t - \frac{t}{n\omega}\cos n\omega t\right\}_0^{\tau/4 = \frac{\pi}{2\omega}} - \frac{4A}{\tau}\left\{\frac{1}{n^2\omega^2}\sin n\omega t - \frac{t}{n\omega}\cos n\omega t\right\}_{\frac{\tau}{4}=\frac{\pi}{2\omega}}^{\frac{3\tau}{4}=\frac{3\pi}{2\omega}} + 2A\left(-\frac{\cos n\omega t}{n\omega}\right)_{\frac{\tau}{4}=\frac{\pi}{2\omega}}^{\frac{3\tau}{4}=\frac{3\pi}{2\omega}} + \frac{4A}{\tau}\left\{\frac{1}{n^2\omega^2}\sin n\omega t\right.\right.$$

$$- \frac{t}{n\omega} \cos n\omega t \Big|_{\frac{3\tau}{4} = \frac{3\pi}{2\omega}}^{\tau = \frac{2\pi}{\omega}} - 4A \left(- \frac{\cos n\omega t}{n\omega} \right)_{\frac{3\tau}{4} = \frac{3\pi}{2\omega}}^{\tau = \frac{2\pi}{\omega}} \Bigg]$$

$$= \frac{4A}{\pi^2 n^2} \left(\sin \frac{n\pi}{2} - \sin \frac{3n\pi}{2} \right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8A}{\pi^2 n^2} (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \ x(t) = \frac{8A}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} (-1)^{\frac{n-1}{2}} \frac{\sin n\omega t}{n^2}$$

(1.65) $x(t) = A\left(1 - \frac{t}{\tau}\right) \quad , \quad 0 \le t \le \tau$

$$a_0 = \frac{2}{\tau} \int_0^\tau x(t)\, dt = \frac{2A}{\tau} \int_0^\tau \left(1 - \frac{t}{\tau}\right) dt = \frac{2A}{\tau} \left(t - \frac{t^2}{2\tau} \right)_0^\tau = A$$

$$a_n = \frac{2}{\tau} \int_0^\tau x(t) \cos n\omega t\, dt = \frac{2A}{\tau} \left(\frac{\sin n\omega t}{n\omega} - \frac{t}{\tau} \frac{\sin n\omega t}{n\omega} - \frac{\cos n\omega t}{\tau n^2 \omega^2} \right)_0^{2\pi/\omega}$$

$$= 0$$

$$b_n = \frac{2}{\tau} \int_0^\tau x(t) \sin n\omega t\, dt = \frac{2A}{\tau} \left(- \frac{\cos n\omega t}{n\omega} + \frac{t}{\tau} \frac{\cos n\omega t}{n\omega} - \frac{\sin n\omega t}{\tau n^2 \omega^2} \right)_0^{2\pi/\omega}$$

$$= \frac{A}{\pi n}$$

$$\therefore \ x(t) = \frac{A}{2} + \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\omega t}{n}$$

(1.66) The truncated series of k terms can be denoted as

$$\overline{x}(t) = \frac{\overline{a}_0}{2} + \sum_{n=1}^{k} \overline{a}_n \cos n\,\omega\, t + \sum_{n=1}^{k} \overline{b}_n \sin n\,\omega\, t \tag{1}$$

with $\overline{x}(t)$ denoting an approximation to the exact $x(t)$ given by Eq. (1.70). The error to be minimized is given by

$$E = \int_{-\pi/\omega}^{\pi/\omega} e^2(t)\, dt \tag{2}$$

$$\text{where} \quad e(t) = x(t) - \overline{x}(t) \tag{3}$$

and $x(t)$ is the exact value (with infinite series on the right hand side of Eq. (1)). Treating E as a function of the unknowns $\overline{a}_n$ and $\overline{b}_n$, it can be minimized by setting:

$$\frac{\partial E}{\partial \overline{a}_n} = 2 \int_{-\pi/\omega}^{\pi/\omega} \left\{ x(t) - \overline{x}(t) \right\} \left(- \cos n\,\omega\, t \right) dt = 0 \tag{4}$$

$$\frac{\partial E}{\partial \overline{b}_n} = 2 \int_{-\pi/\omega}^{\pi/\omega} \left\{ x(t) - \overline{x}(t) \right\} \left(- \sin n\,\omega\, t \right) dt = 0 \tag{5}$$

Rearranging Eq. (4) gives

$$\int_{-\pi/\omega}^{\pi/\omega} x(t) \cos n\,\omega\, t\, dt = \int_{-\pi/\omega}^{\pi/\omega} \overline{x}(t) \cos n\,\omega\, t\, dt \tag{6}$$

Using orthogonalty property, the right hand side of Eq. (6) can be expressed as

$$\int_{-\pi/\omega}^{\pi/\omega} \overline{x}(t) \cos n\,\omega\, t\, dt = \begin{cases} 0 & \text{for } m \neq n \\ \dfrac{\overline{a}_n\, \pi}{\omega} & \text{for } m = n \end{cases} \tag{7}$$

This leads to

$$\int_{-\pi/\omega}^{\pi/\omega} x(t) \cos n\,\omega\, t\, dt = \frac{\overline{a}_n\, \pi}{\omega} \tag{8}$$

$$\text{or} \quad \overline{a}_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} x(t) \cos n\,\omega\, t\, dt \quad ; \quad n = 0, 1, 2, \ldots, k \tag{9}$$

In a similar manner, we can derive:

$$\overline{b}_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} x(t) \sin n\,\omega\, t\, dt \quad ; \quad n = 1, 2, \ldots, k \tag{10}$$

It can be observed that Eqs. (9) and (10) are simlar to those of Eqs. (E.3) and (E.4).

1.67

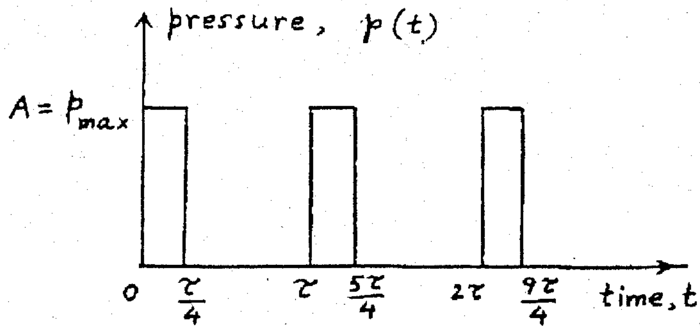
i	t_i	x_i	$n=1$: $x_i \cos \frac{2\pi t_i}{0.32}$	$n=1$: $x_i \sin \frac{2\pi t_i}{0.32}$	$n=2$: $x_i \cos \frac{4\pi t_i}{0.32}$	$n=2$: $x_i \sin \frac{4\pi t_i}{0.32}$	$n=3$: $x_i \cos \frac{6\pi t_i}{0.32}$	$n=3$: $x_i \sin \frac{6\pi t_i}{0.32}$
1	0.02	9	8.3149	3.4442	6.3639	6.3640	3.4441	8.3149
2	0.04	13	9.1924	9.1924	0.0000	13.0000	−9.1924	9.1923
3	0.06	17	6.5056	15.7060	−12.0209	12.0208	−15.7059	−6.5057
4	0.08	29	0.0000	29.0000	−29.0000	0.0000	0.0000	−29.0000
5	0.10	43	−16.4556	39.7267	−30.4053	−30.4059	39.7271	−16.4548
6	0.12	59	−41.7195	41.7191	0.0000	−59.0000	41.7187	41.7199
7	0.14	63	−58.2045	24.1087	44.5482	−44.5472	−24.1101	58.2040
8	0.16	57	−57.0000	0.0000	57.0000	0.0000	−57.0000	0.0000
9	0.18	49	−45.2700	−18.7518	34.6477	34.6487	−18.7505	−45.2705
10	0.20	35	−24.7485	−24.7489	0.0000	35.0000	24.7493	−24.7482
11	0.22	35	−13.3936	−32.3359	−24.7493	24.7482	32.3354	13.3950
12	0.24	41	0.0000	−41.0000	−41.0000	0.0000	0.0000	41.0000
13	0.26	47	17.9866	−43.4221	−33.2333	−33.2347	−43.4229	17.9847
14	0.28	41	28.9917	−28.9911	0.0000	−41.0000	−28.9905	−28.9923
15	0.30	13	12.0105	−4.9747	9.1927	−9.1921	4.9755	−12.0102
16	0.32	7	7.0000	0.0000	7.0000	0.0000	7.0000	0.0000

$\sum_{i=1}^{16}(\;)$	558	-166.7897	-31.3278	-11.6552	-91.5984	-43.2234	26.8281
$\frac{1}{8}\sum_{i=1}^{16}(\;)$	69.75	-20.8487	-3.9160	-1.4569	-11.4498	-5.4029	3.3535

1.68 Speed = 100 rpm

In a minute, a point will be subjected to the maximum pressure, $A = p_{max} = 100$ psi, $100 \times 4 = 400$ times. Hence

period $= \tau = \frac{60}{400} = 0.15$ sec.



$$p(t) = \begin{cases} A, & 0 \le t \le \tau/4 \\ 0, & \tau/4 \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^\tau p(t)\,dt = \frac{2}{\tau}A\left(t\right)_0^{\tau/4} = \frac{A}{2} = 50 \text{ psi}$$

$$a_m = \frac{2}{\tau}\int_0^\tau p(t)\cos m\omega t\,dt = \frac{2A}{\tau}\left(\frac{\sin m\omega t}{m\omega}\right)_0^{\tau/4} = \frac{A}{\pi m}\sin\frac{m\pi}{2}$$

$$b_m = \frac{2}{\tau}\int_0^\tau p(t)\sin m\omega t\,dt = -\frac{2A}{\tau}\left(\frac{\cos m\omega t}{m\omega}\right)_0^{\tau/4} = -\frac{A}{\pi m}\left(\cos\frac{m\pi}{2} - 1\right)$$

Evaluation of a_m and b_m:

$m = 1$	$m = 2$	$m = 3$
$a_1 = \frac{A}{\pi}\sin\frac{\pi}{2} = \frac{A}{\pi}$ $= 31.8309$ psi	$a_2 = \frac{A}{2\pi}\sin\pi = 0$	$a_3 = \frac{A}{3\pi}\sin\frac{3\pi}{2}$ $= -10.6103$ psi
$b_1 = -\frac{A}{\pi}\left(\cos\frac{\pi}{2} - 1\right)$ $= 31.8309$ psi	$b_2 = -\frac{A}{2\pi}\left(\cos\pi - 1\right)$ $= 31.8309$ psi	$b_3 = -\frac{A}{3\pi}\left(\cos\frac{3\pi}{2} - 1\right)$ $= 10.6103$ psi

$$\therefore\; p(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\cos m\omega t + b_m\sin m\omega t\right) \quad \text{psi}$$

1.69

i	t_i	M_{t_i}	$n=1$: $M_{t_i}\cos\frac{2\pi t_i}{0.012}$	$n=1$: $M_{t_i}\sin\frac{2\pi t_i}{0.012}$	$n=2$: $M_{t_i}\cos\frac{4\pi t_i}{0.012}$	$n=2$: $M_{t_i}\sin\frac{4\pi t_i}{0.012}$	$n=3$: $M_{t_i}\cos\frac{6\pi t_i}{0.012}$	$n=3$: $M_{t_i}\sin\frac{6\pi t_i}{0.012}$
1	0.0005	770	743.7627	199.2912	666.8391	385.0010	544.4712	544.4731
2	0.0010	810	701.4802	405.0007	404.9988	701.4812	0.0000	810.0000
3	0.0015	850	601.0398	601.0417	0.0000	850.0000	-601.0442	601.0373
4	0.0020	910	454.9978	788.0845	-455.0041	788.0808	-910.0000	0.0000

5	0.0025	1010	261.4043	975.5859	−874.689	504.995	−714.171	−714.184
6	0.0030	1170	0.0000	1170.0000	−1170.000	0.000	0.000	−1170.000
7	0.0035	1370	−354.5874	1323.3169	−1186.449	−685.010	968.748	−968.725
8	0.0040	1610	−805.0073	1394.2966	−804.987	−1394.309	1610.000	0.000
9	0.0045	1890	−1336.4407	1336.4229	0.000	−1890.000	1336.410	1336.454
10	0.0050	1750	−1515.5491	874.9922	875.019	−1515.534	0.000	1750.000
11	0.0055	1630	−1574.4619	421.8647	1411.634	−814.979	−1152.608	1152.560
12	0.0060	1510	−1510.0000	0.0000	1510.000	0.000	−1510.000	0.000
13	0.0065	1390	−1342.6345	−359.7671	1203.767	695.014	−982.858	−982.898
14	0.0070	1290	−1117.1677	−645.0088	644.982	1117.183	0.000	−1290.000
15	0.0075	1190	−841.4492	−841.4648	0.000	1190.000	841.479	−841.435
16	0.0080	1110	−554.9897	−961.2942	−555.021	961.276	1110.000	0.000
17	0.0085	1050	−271.7498	−1014.2249	−909.337	524.982	742.440	742.485
18	0.0090	990	0.0000	−990.0000	−990.000	0.000	0.000	990.000
19	0.0095	930	240.7123	−898.3081	−805.393	−465.018	−657.633	657.586
20	0.0100	890	445.0095	−770.7571	−444.981	−770.773	−890.000	0.000
21	0.0105	850	601.0478	−601.0337	0.000	−850.000	−601.022	−601.060
22	0.0110	810	701.4868	−404.9895	405.022	−701.468	0.000	−810.000
23	0.0115	770	743.7659	−199.2798	666.851	−384.980	544.500	−544.444
24	0.0120	750	750.0000	0.0000	750.000	0.000	750.000	0.000
$\sum_{i=1}^{24}(\ )$		27,300	−4,979.3242	1,803.7673	343.270	−1,754.047	428.734	661.855
$\frac{1}{12}\sum_{i=1}^{24}$		2,275	−414.9436	150.3139	28.606	−146.171	35.728	55.155

1.70

i	t_i	x_i	$n=1$: $x_i \cos\frac{2\pi t_i}{0.6}$	$n=1$: $x_i \sin\frac{2\pi t_i}{0.6}$	$n=2$: $x_i \cos\frac{4\pi t_i}{0.6}$	$n=2$: $x_i \sin\frac{4\pi t_i}{0.6}$	$n=3$: $x_i \cos\frac{6\pi t_i}{0.6}$	$n=3$: $x_i \sin\frac{6\pi t_i}{0.6}$
1	0.025	9.00	8.69	2.33	7.79	4.50	6.36	6.36
2	0.050	17.00	14.72	8.50	8.50	14.72	0.00	17.00
3	0.075	23.00	16.26	16.26	0.00	23.00	−16.26	16.26
4	0.100	25.00	12.50	21.65	−12.50	21.65	−25.00	0.00
5	0.125	26.00	6.73	25.11	−22.52	13.00	−18.38	−18.38
6	0.150	28.00	0.00	28.00	−28.00	0.00	0.00	−28.00
7	0.175	33.00	−8.54	31.88	−28.58	−16.50	23.33	−23.33
8	0.200	35.00	−17.50	30.31	−17.50	−30.31	35.00	0.00
9	0.225	34.00	−24.04	24.04	0.00	−34.00	24.04	24.04
10	0.250	29.00	−25.11	14.50	14.50	−25.11	0.00	29.00
11	0. 75	24.00	−23.18	6.21	20.78	−12.00	−16.97	16.97
12	0.300	26.00	−26.00	0.00	26.00	0.00	−26.00	0.00
13	0.325	32.00	−30.91	−8.28	27.71	16.00	−22.63	−22.63
14	0.350	40.00	−34.64	−20.00	20.00	34.64	0.00	−40.00
15	0.375	18.00	−12.73	−12.73	0.00	18.00	12.73	−12.73
16	0.400	8.00	−4.00	−6.93	−4.00	6.93	8.00	0.00
17	0.425	−5.00	1.29	4.83	4.33	−2.50	−3.54	−3.54
18	0.450	−14.00	0.00	14.00	14.00	0.00	0.00	−14.00
19	0.475	−28.00	−7.25	27.05	24.25	14.00	19.80	−19.80
20	0.500	−37.00	−18.50	32.04	18.50	32.04	37.00	0.00
21	0.525	−33.00	−23.33	23.33	0.00	33.00	23.33	23.34
22	0.550	−29.00	−25.11	14.50	−14.50	25.11	0.00	29.00

23	0.575	-22.00	-21.25	5.69	-19.05	11.00	-15.56	15.56
24	0.600	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\sum_{i=1}^{24}(\ )$		239.00	-241.90	282.30	39.72	147.18	45.26	-4.88
$\frac{1}{12}\sum_{i=1}^{24}(\ )$		19.92	-20.16	23.53	3.31	12.26	3.77	-0.41

(1.71) *The main program and the output are shown below.*

```
C ===========================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE FORIER
C
C ===========================================================================
C FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION X(24),T(24),XSIN(24),XCOS(24),A(5),B(5)
      DATA N,M,TIME /24,5,0.012/
      DATA X/770,810,850,910,1010,1170,1370,1610,1890,1750,1630,1510,
     2 1390,1290,1190,1110,1050,990,930,890,850,810,770,750/
      DATA T /.0005,.001,.0015,.002,.0025,.003,.0035,.004,.0045,.005,
     2 .0055,.006,.0065,.007,.0075,.008,.0085,.009,.0095,.01,.0105,
     3 .011,.0115,.012/
C END OF PROBLEM-DEPENDENT DATA
      CALL FORIER (N,M,TIME,X,T,AZERO,A,B,XSIN,XCOS)
      PRINT 100
 100  FORMAT (//,46H FOURIER SERIES EXPANSION OF THE FUNCTION X(T),//)
      PRINT 200, N,M,TIME
 200  FORMAT (6H DATA:,//,37H NUMBER OF DATA POINTS IN ONE CYCLE =,I5,
     2 /,42H NUMBER OF FOURIER COEFFICIENTS REQUIRED =,I5,/,
     3 14H TIME PERIOD =,E15.8)
      PRINT 300, (T(I),I=1,N)
 300  FORMAT (/,33H TIME AT VARIOUS STATIONS, T(I) =,/,(4E15.8,1X))
      PRINT 400, (X(I),I=1,N)
 400  FORMAT (/,31H KNOWN VALUES OF X(I) AT T(I) =,/,(4E15.8,1X))
      PRINT 500
 500  FORMAT (//,29H RESULTS OF FOURIER ANALYSIS:,/)
      PRINT 600, AZERO
 600  FORMAT (8H AZERO =,2X,E15.8,//,31H VALUES OF I, A(I) AND B(I) ARE
     2 ,/)
      DO 700, I =1,M
 700  PRINT 800, I, A(I), B(I)
 800  FORMAT (I5,2X,E15.8,2X,E15.8)
      STOP
      END
```

```
FOURIER SERIES EXPANSION OF THE FUNCTION X(T)


DATA:

NUMBER OF DATA POINTS IN ONE CYCLE =    24
NUMBER OF FOURIER COEFFICIENTS REQUIRED =     5
TIME PERIOD = 0.11999998E-01
```

```
TIME AT VARIOUS STATIONS, T(I) =
0.50000008E-03 0.99999993E-03 0.15000000E-02 0.20000001E-02
0.24999999E-02 0.30000000E-02 0.35000001E-02 0.40000007E-02
0.45000017E-02 0.49999990E-02 0.55000000E-02 0.60000010E-02
0.64999983E-02 0.69999993E-02 0.75000003E-02 0.80000013E-02
0.84999986E-02 0.89999996E-02 0.95000006E-02 0.10000002E-01
0.10499999E-01 0.11000000E-01 0.11500001E-01 0.11999998E-01

KNOWN VALUES OF X(I) AT T(I) =
0.77000000E+03 0.81000000E+03 0.85000000E+03 0.91000000E+03
0.10100000E+04 0.11700000E+04 0.13700000E+04 0.16100000E+04
0.18900000E+04 0.17500000E+04 0.16300000E+04 0.15100000E+04
0.13900000E+04 0.12900000E+04 0.11900000E+04 0.11100000E+04
0.10500000E+04 0.99000000E+03 0.93000000E+03 0.89000000E+03
0.85000000E+03 0.81000000E+03 0.77000000E+03 0.75000000E+03


RESULTS OF FOURIER ANALYSIS:

AZERO =    0.22750000E+04

VALUES OF I, A(I) AND B(I) ARE

    1  -0.41494360E+03   0.15031395E+03
    2   0.28605835E+02  -0.14617058E+03
    3   0.35727844E+02   0.55154602E+02
    4  -0.40830078E+02   0.14440117E+01
    5   0.11577332E+02  -0.16659973E+02
```

1.72 *The main program and the output are given below.*

```
C ====================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE FORIER
C
C ====================================================================
C FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION X(16),T(16),XSIN(16),XCOS(16),A(5),B(5)
      DATA N,M,TIME /16,5,0.32/
      DATA X /9.,13.,17.,29.,43.,59.,63.,57.,49.,35.,35.,41.,47.,41.,
     2 13.,7./
      DATA T /.02,.04,.06,.08,.1,.12,.14,.16,.18,.2,.22,.24,.26,.28,
     2 .3,.32/
C END OF PROBLEM-DEPENDENT DATA
      CALL FORIER (N,M,TIME,X,T,AZERO,A,B,XSIN,XCOS)
      PRINT 100
 100  FORMAT (//,46H FOURIER SERIES EXPANSION OF THE FUNCTION X(T),//)
      PRINT 200, N,M,TIME
 200  FORMAT (6H DATA:,//,37H NUMBER OF DATA POINTS IN ONE CYCLE =,I5,
     2 /,42H NUMBER OF FOURIER COEFFICIENTS REQUIRED =,I5,/,
     3 14H TIME PERIOD =,E15.8)
      PRINT 300, (T(I),I=1,N)
 300  FORMAT (/,33H TIME AT VARIOUS STATIONS, T(I) =,/,(4E15.8,1X))
      PRINT 400, (X(I),I=1,N)
 400  FORMAT (/,31H KNOWN VALUES OF X(I) AT T(I) =,/,(4E15.8,1X))
      PRINT 500
```

```
 500  FORMAT (//,29H RESULTS OF FOURIER ANALYSIS:,/)
      PRINT 600, AZERO
 600  FORMAT (8H AZERO =,2X,E15.8,//,31H VALUES OF I, A(I) AND B(I) ARE
     2 ,/)
      DO 700, I =1,M
 700  PRINT 800, I, A(I), B(I)
 800  FORMAT (I5,2X,E15.8,2X,E15.8)
      STOP
      END
```

```
FOURIER SERIES EXPANSION OF THE FUNCTION X(T)


DATA:

NUMBER OF DATA POINTS IN ONE CYCLE =    16
NUMBER OF FOURIER COEFFICIENTS REQUIRED =     5
TIME PERIOD = 0.31999999E+00

TIME AT VARIOUS STATIONS, T(I) =
0.20000000E-01 0.39999999E-01 0.59999999E-01 0.79999983E-01
0.10000002E+00 0.12000000E+00 0.13999999E+00 0.16000003E+00
0.18000001E+00 0.19999999E+00 0.22000003E+00 0.24000001E+00
0.25999999E+00 0.27999997E+00 0.30000001E+00 0.31999999E+00

KNOWN VALUES OF X(I) AT T(I) =
0.90000000E+01 0.13000000E+02 0.17000000E+02 0.29000000E+02
0.43000000E+02 0.59000000E+02 0.63000000E+02 0.57000000E+02
0.49000000E+02 0.35000000E+02 0.35000000E+02 0.41000000E+02
0.47000000E+02 0.41000000E+02 0.13000000E+02 0.70000000E+01


RESULTS OF FOURIER ANALYSIS:

AZERO =    0.69750000E+02

VALUES OF I, A(I) AND B(I) ARE

    1  -0.20848709E+02  -0.39159737E+01
    2  -0.14568996E+01  -0.11449797E+02
    3  -0.54029312E+01   0.33535175E+01
    4  -0.17500381E+01   0.24999523E+01
    5  -0.26129246E-01   0.10607624E+01
```

(1.73) *The main program and the output are shown below.*

```
C =====================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE FORIER
C
C =====================================================================
C FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION X(24),T(24),XSIN(24),XCOS(24),A(6),B(6)
      DATA N,M,TIME /24,6,0.6/
      DATA X/9,17,23,25,26,28,33,35,34,29,24,26,32,40,18,8,-5,-14,
     2 -28,-37,-33,-29,-22,0/
```

```
      DATA T /.025,.05,.075,.1,.125,.15,.175,.2,.225,.25,.275,.3,.325,
     2 .35,.375,.4,.425,.45,.475,.5,.525,.55,.575,.6/
C END OF PROBLEM-DEPENDENT DATA
      CALL FORIER (N,M,TIME,X,T,AZERO,A,B,XSIN,XCOS)
      PRINT 100
 100  FORMAT (//,46H FOURIER SERIES EXPANSION OF THE FUNCTION X(T),//)
      PRINT 200, N,M,TIME
 200  FORMAT (6H DATA:,//,37H NUMBER OF DATA POINTS IN ONE CYCLE =,I5,
     2 /,42H NUMBER OF FOURIER COEFFICIENTS REQUIRED =,I5,/,
     3 14H TIME PERIOD =,E15.8)
      PRINT 300, (T(I),I=1,N)
 300  FORMAT (/,33H TIME AT VARIOUS STATIONS, T(I) =,/,(4E15.8,1X))
      PRINT 400, (X(I),I=1,N)
 400  FORMAT (/,31H KNOWN VALUES OF X(I) AT T(I) =,/,(4E15.8,1X))
      PRINT 500
 500  FORMAT (//,29H RESULTS OF FOURIER ANALYSIS:,/)
      PRINT 600, AZERO
 600  FORMAT (8H AZERO =,2X,E15.8,//,31H VALUES OF I, A(I) AND B(I) ARE
     2 ,/)
      DO 700, I =1,M
 700  PRINT 800, I, A(I), B(I)
 800  FORMAT (I5,2X,E15.8,2X,E15.8)
      STOP
      END
```

```
 FOURIER SERIES EXPANSION OF THE FUNCTION X(T)


 DATA:

 NUMBER OF DATA POINTS IN ONE CYCLE =   24
 NUMBER OF FOURIER COEFFICIENTS REQUIRED =    6
 TIME PERIOD = 0.60000002E+00

 TIME AT VARIOUS STATIONS, T(I) =
 0.24999999E-01 0.50000001E-01 0.74999988E-01 0.10000002E+00
 0.12500000E+00 0.14999998E+00 0.17500001E+00 0.19999999E+00
 0.22500002E+00 0.25000000E+00 0.27499998E+00 0.30000001E+00
 0.32499999E+00 0.35000002E+00 0.37500000E+00 0.39999998E+00
 0.42500001E+00 0.44999999E+00 0.47500002E+00 0.50000000E+00
 0.52499998E+00 0.55000001E+00 0.57499999E+00 0.60000002E+00

 KNOWN VALUES OF X(I) AT T(I) =

 0.90000000E+01 0.17000000E+02 0.23000000E+02 0.25000000E+02
 0.26000000E+02 0.28000000E+02 0.33000000E+02 0.35000000E+02
 0.34000000E+02 0.29000000E+02 0.24000000E+02 0.26000000E+02
 0.32000000E+02 0.40000000E+02 0.18000000E+02 0.80000000E+01
-0.50000000E+01-0.14000000E+02-0.28000000E+02-0.37000000E+02
-0.33000000E+02-0.29000000E+02-0.22000000E+02 0.00000000E+00


 RESULTS OF FOURIER ANALYSIS:

 AZERO =   0.19916672E+02
```

```
VALUES OF I, A(I) AND B(I) ARE

    1  -0.20158676E+02   0.23525284E+02
    2   0.33099222E+01   0.12264638E+02
    3   0.37719278E+01  -0.40640426E+00
    4  -0.95843577E+00   0.32476425E+01
    5   0.11377630E+01  -0.18716125E+01
    6  -0.11667604E+01   0.12500324E+01
```

(1.74) *The main program and the output are given below.*

```
C =====================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE FORIER
C
C =====================================================================
C FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION X(14),T(14),XSIN(14),XCOS(14),A(10),B(10)
      DATA N,M,TIME /14,10,0.35/
      DATA X /.45,-.8,-.9,-.6,-.75,-.7,.55,1.75,1.65,.25,-1.1,
     2 -1.4,-1.05,0.0/
      DATA T /.025,.05,.075,.1,.125,.15,.175,.2,.225,.25,.275,.3,.325,
     2 .35/
C END OF PROBLEM-DEPENDENT DATA
      CALL FORIER (N,M,TIME,X,T,AZERO,A,B,XSIN,XCOS)
      PRINT 100
 100  FORMAT (//,46H FOURIER SERIES EXPANSION OF THE FUNCTION X(T),//)
      PRINT 200, N,M,TIME
 200  FORMAT (6H DATA:,//,37H NUMBER OF DATA POINTS IN ONE CYCLE =,I5,
     2 /,42H NUMBER OF FOURIER COEFFICIENTS REQUIRED =,I5,/,
     3 14H TIME PERIOD =,E15.8)
      PRINT 300, (T(I),I=1,N)
 300  FORMAT (/,33H TIME AT VARIOUS STATIONS, T(I) =,/,(4E15.8,1X))
      PRINT 400, (X(I),I=1,N)
 400  FORMAT (/,31H KNOWN VALUES OF X(I) AT T(I) =,/,(4E15.8,1X))
      PRINT 500
 500  FORMAT (//,29H RESULTS OF FOURIER ANALYSIS:,/)
      PRINT 600, AZERO
 600  FORMAT (8H AZERO =,2X,E15.8,//,31H VALUES OF I, A(I) AND B(I) ARE
     2 ,/)
      DO 700, I =1,M
 700  PRINT 800, I, A(I), B(I)
 800  FORMAT (I5,2X,E15.8,2X,E15.8)
      STOP
      END
```

```
FOURIER SERIES EXPANSION OF THE FUNCTION X(T)


DATA:

NUMBER OF DATA POINTS IN ONE CYCLE =    14
NUMBER OF FOURIER COEFFICIENTS REQUIRED =    10
TIME PERIOD = 0.35000002E+00
```

```
TIME AT VARIOUS STATIONS, T(I) =
0.24999999E-01 0.50000001E-01 0.74999988E-01 0.10000002E+00
0.12500000E+00 0.14999998E+00 0.17500001E+00 0.19999999E+00
0.22500002E+00 0.25000000E+00 0.27499998E+00 0.30000001E+00
0.32499999E+00 0.35000002E+00

KNOWN VALUES OF X(I) AT T(I) =
 0.44999999E+00-0.80000001E+00-0.89999998E+00-0.60000002E+00
-0.75000000E+00-0.69999999E+00 0.55000001E+00 0.17500000E+01
 0.16499996E+01 0.25000000E+00-0.11000004E+01-0.13999996E+01
-0.10500002E+01 0.00000000E+00


RESULTS OF FOURIER ANALYSIS:

AZERO =   -0.37857169E+00

VALUES OF I, A(I) AND B(I) ARE

    1   -0.61951119E+00   -0.35046142E+00
    2    0.46242946E+00    0.92408919E+00
    3    0.41494656E+00   -0.17127156E+00
    4    0.22273250E-01    0.24691588E+00
    5   -0.45435190E-01    0.10430527E+00
    6   -0.20434089E-01    0.56009147E-01
    7   -0.49999803E-01   -0.35824633E-05
    8   -0.20436604E-01   -0.55996872E-01
    9   -0.45427930E-01   -0.10429442E+00
   10    0.22274703E-01   -0.24690533E+00
```

1.75 $x_p = r + l - r\cos\theta - l\cos\phi = r + l - r\cos\omega t - l\sqrt{1-\sin^2\phi}$ $\quad (E_1)$

But $l\sin\phi = r\sin\theta$, $\quad \cos\phi = \left(1 - \frac{r^2}{l^2}\sin^2\omega t\right)^{\frac{1}{2}}$ $\quad (E_2)$

Using (E_2) in (E_1), $\quad x_p = r + l - r\cos\omega t - l\left(1 - \frac{r^2}{l^2}\sin^2\omega t\right)^{\frac{1}{2}}$ $\quad (E_3)$

Let $\frac{r}{l}$ = small $\left(< \frac{1}{4}\right)$. Using $\sqrt{1-\epsilon} \approx 1 - \frac{1}{2}\epsilon$, (E_3) becomes

$$x_p \approx r\left(1 + \frac{r}{2l}\right) - r\left(\cos\omega t + \frac{r}{4l}\cos 2\omega t\right) \quad (E_4)$$

(a) Eq. (E_4) gives

$$y_p = x_p - r\left(1 + \frac{r}{2l}\right) \simeq -r\left(\cos\omega t + \frac{1}{4}\frac{r}{l}\cos 2\omega t\right) \quad \text{---}\ (E_5)$$

If $\frac{r}{l}$ is very small, $y_p \simeq -r\cos\omega t \Rightarrow$ harmonic motion.

(b) To have amplitude of second harmonic smaller than that of first harmonic in Eq. (E_5), we need to have

$$\frac{1}{4}\frac{r}{l} \le \frac{1}{25}, \quad \text{i.e.,} \quad \frac{r}{l} \le \frac{4}{25}, \quad \text{i.e.,} \quad \frac{l}{r} \ge 6.25$$

Once the amplitude of second harmonic is smaller by a factor of 25, the amplitudes of higher harmonics arising from the expansion of square-root-term in (E_3) are expected to be still smaller.

1.76 Unbalanced force developed = P = $2\,m\,\omega^2\,r\cos\omega t$, range of force = 0 - 100 N, range of frequency = 25 - 50 Hz = 157.08 - 314.16 rad/sec.
Parameters to be determined: m, r, ω.
Let r = 0.1 m. To generate 100 N force at 25 Hz, set:

$$P_{max} = 100 = 2\,m\,(157.08)^2\,(0.1)$$

which gives

$$m = \frac{100}{2\,(157.08)^2\,(0.1)} = 0.0202641 \text{ kg} = 20.2641 \text{ g}$$

To generate 100 N force at 50 Hz, set:

$$P_{max} = 100 = 2\,m\,(314.16)^2\,(0.1)$$

which yields

$$m = \frac{100}{2\,(314.16)^2\,(0.1)} = 0.0050660 \text{ kg} = 5.0660 \text{ g}$$

1.77 Goal: Weight to be maintained at 10 ± 0.1 lb/min
Parameters to be determined: Angular velocity of crank (ω), lengths of crank and connecting rod, dimensions of the wedge, dimensions of the orifice in the hopper, dimensions of the actuating rod, and dimensions of the lever arrangement.
Given: Density of the material in the hopper.
Procedure:
Select ω based on available motor. Determine the dimensions of the orifice in the hopper which delivers approximately 10 lb/min (assuming continuous flow of material). For trial dimensions of the wedge, determine the increase/decrease in the size (diameter) of the orifice. Choose the final dimensions of the wedge such that the material flow rate delivered by the orifice lies within the specified range.

1.78 Force to be applied = 200 lb, frequency = 50 Hz = 314.16 rad/sec.

Procedure:

1. Select a motor that provides, either directly or through a gear system, the desired frequency. Assume that it is connected to the cam.
2. Setermine the sizes and dimensions of the plate cam and the roller.
3. Choose the dimensions of the follower.
4. Select the weight as 200 lb. From the geometry, determine the range of displacement (vertical motion) of the weight.
5. Determine the force exerted due to the falling weight.

1.79 Considerations to be taken in the design of vibratory bowl feerders:

1. Suitable design of the electromagnet and its coil.
2. Radius of the bowl and the pitch of the spiral (helical) delivery track.
3. Tooling to be fixed along the spiral track to reject the defective or out-of-tolerance or incorrectly oriented parts.
4. Design of elastic supports.
5. Size and location of the outlet.

1.80 Axial spring constant of each tube $= k = \frac{A\,E}{\ell}$.

Let diameter of each tube be 0.01 m (1 cm) with thickness 0.001 m (1 mm). Then

$$A = \frac{\pi}{4}\,(D^2 - d^2) = \frac{\pi}{4}\,(0.01^2 - 0.008^2) = 28.27\,(10^{-6})\ m^2$$

This gives

$$k = \frac{(28.27\,(10^{-6}))\,(2.07\,(10^{11}))}{2} = 29.26\,(10^5)\ N/m$$

Since 76 tubes are in parallel, we have the total axial stiffness as:

$$k_{eq} = 76\ k = (76)\,(29.26\,(10^5)) = 222.38\,(10^6)\ N/m$$

The polar area moment of inertia of each tube is

$$J = \frac{\pi}{32}\,(D^4 - d^4) = \frac{\pi}{32}\,(0.01^4 - 0.008^4) = 580\,(10^{-8})\ m^4$$

Torsional stiffness of each tube is given by

$$\frac{G\,J}{\ell} = \frac{(79.6154\,(10^9))\,(580\,(10^{-8}))}{2} = 231\,(10^3)\ N\text{–}m/rad$$

For 76 tubes in parallel, equivalent torsional stiffness will be:

$$k_{t_{eq}} = (76)\,(231\,(10^3)) = 17.56\,(10^6)\ N\text{–}m/rad$$

Chapter 2

Free Vibration of Single Degree of Freedom Systems

2.1 $\delta_{st} = 5 \times 10^{-3}$ m

$$\omega_n = \left(\frac{g}{\delta_{st}}\right)^{1/2} = \left(\frac{9.81}{5 \times 10^{-3}}\right)^{1/2} = 44.2945 \text{ rad/sec} = 7.0497 \text{ Hz}$$

2.2 $\tau_n = 0.21 \text{ sec} = 2\pi\sqrt{\frac{m}{k}}$, $\sqrt{m} = 0.21\sqrt{k}/2\pi$

(i) $$(\tau_n)_{new} = \frac{2\pi\sqrt{m}}{\sqrt{k_{new}}} = \frac{2\pi\sqrt{m}}{\sqrt{1.5\,k}} = \frac{2\pi\left(\frac{0.21\sqrt{k}}{2\pi}\right)}{\sqrt{1.5\,k}} = 0.1715 \text{ sec.}$$

(ii) $$(\tau_n)_{new} = \frac{2\pi\sqrt{m}}{\sqrt{k_{new}}} = \frac{2\pi\sqrt{m}}{\sqrt{0.5\,k}} = 2\pi\left(\frac{0.21\sqrt{k}}{2\pi}\right)\frac{1}{\sqrt{0.5\,k}} = 0.2970 \text{ sec.}$$

2.3 $\omega_n = 62.832 \text{ rad/sec} = \sqrt{\frac{k}{m}}$, $\sqrt{m} = \sqrt{k}/62.832$

When spring constant is reduced, ω_n decreases.

$$(\omega_n)_{new} = 0.55\,\omega_n = 34.5576 \text{ rad/sec} = \sqrt{\frac{k_{new}}{m_{new}}} = \sqrt{\frac{k-800}{m}}$$

$$\sqrt{\frac{k-800}{k}} \times 62.836 = 34.5576 \quad , \quad \sqrt{\frac{k-800}{k}} = 0.55$$

$$\frac{k-800}{k} = (0.55)^2 = 0.3025$$

$$k = 1146.9534 \text{ N/m}$$

$$\sqrt{m} = \sqrt{k}/62.832 \; ; \qquad m = k/62.832^2 = \frac{1146.9534}{3947.8602}$$

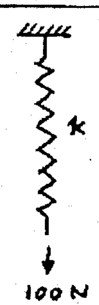
$$m = 0.2905 \text{ kg}$$

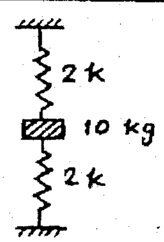
2.4 $k = 100/\left(\frac{10}{1000}\right) = 10000$ N/m

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{4k}{m}} = \left(\frac{4 \times 10^4}{10}\right)^{1/2}$$

$$= 63.2456 \text{ rad/sec}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{6.2832}{63.2456} = 0.0993 \text{ sec}$$





(2.5) $m = \frac{2000}{386.4}$.

Let $\omega_n = 7.5$ rad/sec.

$$\omega_n = \sqrt{\frac{k_{eq}}{m}}$$

$$k_{eq} = m\,\omega_n^2 = \left(\frac{2000}{386.4}\right)(7.5)^2 = 291.1491 \text{ lb/in} = 4\text{ k}$$

where k is the stiffness of the air spring.

Thus $k = \frac{291.1491}{4} = 72.7873$ lb/in.

(2.6) $x = A\cos(\omega_n t - \phi_0)$, $\dot{x} = -\omega_n A \sin(\omega_n t - \phi_0)$,

$\ddot{x} = -\omega_n^2 A \cos(\omega_n t - \phi_0)$

(a) $\omega_n A = 0.1$ m/sec ; $\tau_n = \frac{2\pi}{\omega_n} = 2$ sec, $\omega_n = 3.1416$ rad/sec

$A = 0.1/\omega_n = 0.03183$ m

(d) $x_0 = x(t=0) = A\cos(-\phi_0) = 0.02$ m

$\cos(-\phi_0) = \frac{0.02}{A} = 0.6283$

$\phi_0 = 51.0724°$

(b) $\dot{x}_0 = \dot{x}(t=0) = -\omega_n A \sin(-\phi_0) = -0.1\sin(-51.0724°)$

$= 0.07779$ m/sec

(c) $\ddot{x}\big|_{max} = \omega_n^2 A = (3.1416)^2(0.03183) = 0.314151$ m/sec^2

(2.7) For small angular rotation of bar PQ about P,

$$\frac{1}{2}(k_{12})_{eq}(\theta l_3)^2 = \frac{1}{2}k_1(\theta l_1)^2 + \frac{1}{2}k_2(\theta l_2)^2$$

i.e., $(k_{12})_{eq} = (k_1 l_1^2 + k_2 l_2^2)/l_3^2$

Let k_{eq} = overall spring constant at Q.

$$\frac{1}{k_{eq}} = \frac{1}{(k_{12})_{eq}} + \frac{1}{k_3}$$

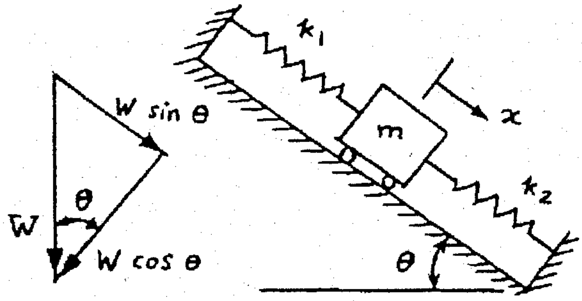
$$k_{eq} = \frac{(k_{12})_{eq}\,k_3}{(k_{12})_{eq} + k_3} = \frac{\left\{k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2\right\}k_3}{k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2 + k_3}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_1 k_2 \ell_1^2 + k_2 k_3 \ell_2^2}{m\,(k_1 \ell_1^2 + k_2 \ell_2^2 + k_3 \ell_3^2)}}$$

2.8 $m = 2000$ kg , $\delta_{st} = 0.02$ m

$$\omega_n = (g/\delta_{st})^{1/2} = \left(\frac{9.81}{0.02}\right)^{1/2} = 22.1472 \text{ rad/sec}$$

2.9 Let x be measured from the position of mass at which the springs are unstretched.



Equation of motion is

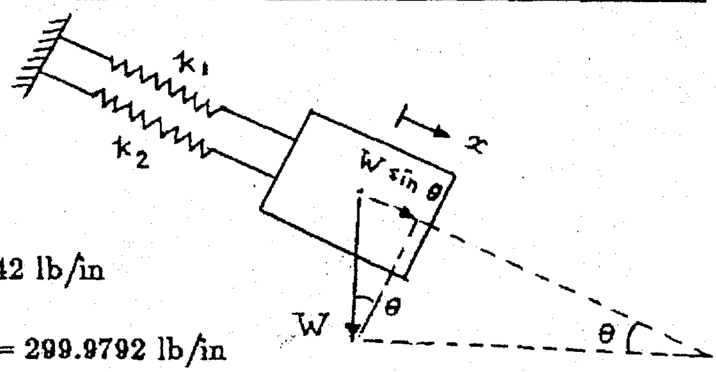
$$m\ddot{x} = -k_1(x+\delta_{st}) - k_2(x+\delta_{st}) + W\sin\theta \quad \text{---- } (E_1)$$

where $\delta_{st}(k_1+k_2) = W\sin\theta$.

Thus Eq. (E_1) becomes $m\ddot{x} + (k_1+k_2)\,x = 0 \Rightarrow \omega_n = \sqrt{\frac{k_1+k_2}{m}}$.

2.10

$$k_1 = \frac{A_1 E_1}{\ell_1} = \frac{\frac{\pi}{4}(0.05)^2\,(30\,(10^6))}{30\,(12)} = 163.6250 \text{ lb/in}$$



$$k_2 = \frac{A_2 E_2}{\ell_2} = \frac{163.625\,(25)}{30} = 136.3542 \text{ lb/in}$$

$$k_{eq} = k_1 + k_2 = 163.6250 + 136.3542 = 299.9792 \text{ lb/in}$$

Let x be measured from the unstretched length of the springs. The equation of motion is:

$$m\ddot{x} = -(k_1+k_2)(x+\delta_{st}) + W\sin\theta$$

where $(k_1+k_2)\,\delta_{st} = W\sin\theta$

$$\text{i.e.,}\quad m\ddot{x} + (k_1+k_2)\,x = 0$$

Thus the natural frequency of vibration of the cart is given by

$$\omega_n = \sqrt{\frac{k_1+k_2}{m}} = \sqrt{\frac{299.9792\,(386.4)}{5000}} = 4.8148 \text{ rad/sec}$$

2.11 Weight of electronic chassis = 500 N. To be able to use the unit in a vibratory environment with a frequency range of 0 - 5 Hz, its natural frequency must be away from the frequency of the environment. Let the natural frequency be $\omega_n = 10$ Hz = 62.832 rad/sec. Since

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = 62.832$$

we have

$$k_{eq} = m\,\omega_n^2 = \left(\frac{500}{9.81}\right)(62.832)^2 = 20.1857\,(10^4)\ \mathrm{N/m} \equiv 4\,k$$

so that k = spring constant of each spring = 50,464.25 N/m. For a helical spring,

$$k = \frac{G\,d^4}{8\,n\,D^3}$$

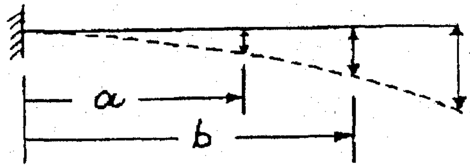
Assuming the material of springs as steel with $G = 80\,(10^9)$ Pa, $n = 5$ and $d = 0.005$ m, we find

$$k = 50{,}464.25 = \frac{80\,(10^9)\,(0.005)^4}{8\,(5)\,D^3}$$

This gives

$$D^3 = \frac{1250\,(10^{-3})}{50464.25} = 24{,}770.0\,(10^{-9}) \quad \text{or} \quad D = 0.0291492 \text{ m} = 2.91492 \text{ cm}$$

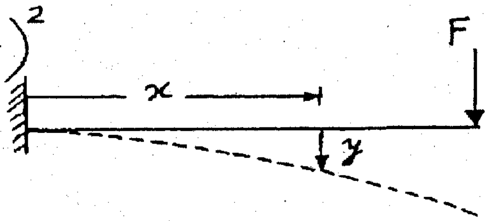
2.12 (i) with springs k_1 and k_2:



Let y_a, y_b, y_ℓ be deflections of beam at distances a, b, ℓ from fixed end.

$$\frac{1}{2}(k_{12})_{eq}\, y_\ell^2 = \frac{1}{2} k_1 y_a^2 + \frac{1}{2} k_2 y_b^2$$

i.e., $$(k_{12})_{eq} = k_1 \left(\frac{y_a}{y_\ell}\right)^2 + k_2 \left(\frac{y_b}{y_\ell}\right)^2$$



$$y = \frac{F x^2}{6EI}(3\ell - x)$$

@ $x = a$, $$y_a = \frac{F a^2}{6EI}(3\ell - a)$$

@ $x = b$, $$y_b = \frac{F b^2}{6EI}(3\ell - b)$$

@ $x = \ell$, $$y_\ell = \frac{F \ell^3}{3EI}$$

$$\omega_n = \left[\frac{k_1 k_3 \left(\frac{y_a}{y_\ell}\right)^2 + k_2 k_3 \left(\frac{y_b}{y_\ell}\right)^2}{m\left\{k_1\left(\frac{y_a}{y_\ell}\right)^2 + k_2\left(\frac{y_b}{y_\ell}\right)^2 + k_{beam}\right\}}\right]^{\frac{1}{2}} \quad \text{where } k_{beam} = \frac{3EI}{\ell^3}$$

$$= \left[\frac{k_1(3EI)\,a^4(3\ell - a)^2 + k_2(3EI)\,b^4(3\ell - b)^2}{m\ell^3\left\{k_1 a^4(3\ell - a)^2 + k_2 b^4(3\ell - b)^2 + 12\,EI\,\ell^3\right\}}\right]^{\frac{1}{2}}$$

(ii) without springs k_1 and k_2:

$$\omega_n = \sqrt{\frac{k_{beam}}{m}} = \sqrt{\frac{3EI}{m\ell^3}}$$

(2.13) Let x_1, x_2 = displacements of pulleys 1, 2

$$x = 2x_1 + 2x_2 \quad \text{----} \quad (E_1)$$

Let P = tension in rope.

For equilibrium of pulley 1,

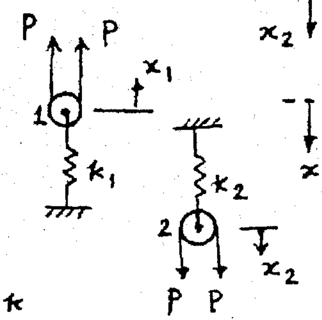
$$2P = k_1 x_1 \quad \text{----} \quad (E_2)$$

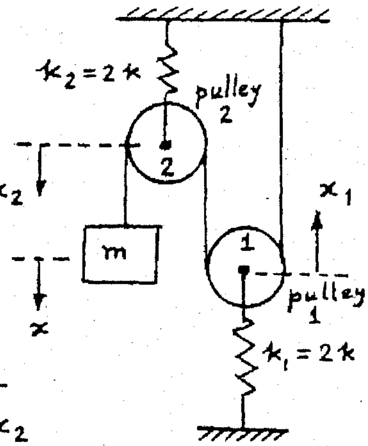
For equilibrium of pulley 2,

$$2P = k_2 x_2 \quad \text{----} \quad (E_3)$$

where $\frac{1}{k_1} = \frac{1}{4k} + \frac{1}{4k} = \frac{1}{2k}$; $k_1 = 2k$

and $k_2 = k + k = 2k$





Combining Eqs. (E_1) to (E_3):

$$x = 2x_1 + 2x_2 = 2\left(\frac{2P}{k_1}\right) + 2\left(\frac{2P}{k_2}\right) = 4P\left(\frac{1}{2k} + \frac{1}{2k}\right) = \frac{4P}{k}$$

Let k_{eq} = equivalent spring constant of the system:

$$k_{eq} = \frac{P}{x} = \frac{k}{4}$$

Equation of motion of mass m: $\quad m\ddot{x} + k_{eq}\, x = 0$

$$\therefore \quad \omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{4m}}$$

(2.14) For a displacement of x of mass m, pulleys 1, 2 and 3 undergo displacements of 2x, 4x and 8x, respectively. The equation of motion of mass m can be written as

$$m\ddot{x} + F_0 = 0 \qquad (1)$$

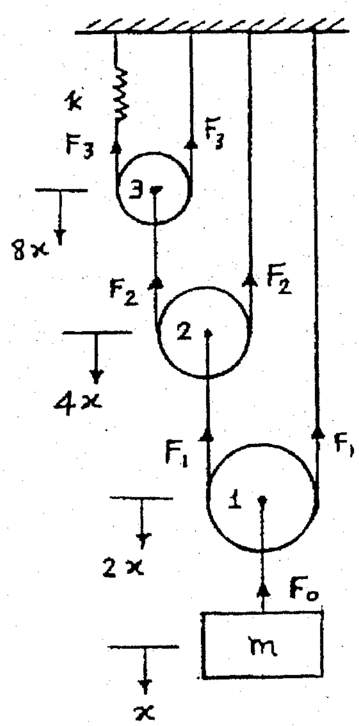
where $F_0 = 2F_1 = 4F_2 = 8F_3$ as shown in figure.

Since $F_3 = (8x)\,k$, Eq. (1) can be rewritten as

$$m\ddot{x} + 8F_3 = 8(8k) = 0 \qquad (2)$$

from which we can find

$$\omega_n = \sqrt{\frac{64\,k}{m}} = 8\sqrt{\frac{k}{m}} \qquad (3)$$



2.15 (a) $\omega_n = \sqrt{4k/M}$

(b) $\omega_n = \sqrt{4k/(M+m)}$

Initial conditions:

Velocity of falling mass $m = v = \sqrt{2gl}$ $\left(\because\ v^2 - \overset{0}{u^2} = 2gl\right)$

$x = 0$ at static equilibrium position.

$x_0 = x(t=0) = -\dfrac{\text{weight}}{k_{eq}} = -\dfrac{mg}{4k}$

Conservation of momentum: $(M+m)\,\dot{x}_0 = m\,v = m\sqrt{2gl}$

$\dot{x}_0 = \dot{x}(t=0) = \dfrac{m}{M+m}\sqrt{2gl}$

Complete solution: $x(t) = A_0 \sin(\omega_n t + \phi_0)$

where $A_0 = \sqrt{x_0^2 + \left(\dfrac{\dot{x}_0}{\omega_n}\right)^2} = \sqrt{\dfrac{m^2 g^2}{16k^2} + \dfrac{m^2 g l}{2k(M+m)}}$

and $\phi_0 = \tan^{-1}\left(\dfrac{x_0 \omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(\dfrac{-\sqrt{g}}{\sqrt{8lk(M+m)}}\right)$

2.16 (a) Velocity of anvil $= v = 50$ ft/sec $= 600$ in/sec. $x = 0$ at static equilibrium position.

$$x_0 = x(t{=}0) = -\frac{\text{weight}}{k_{eq}} = -\frac{m\,g}{4\,k}$$

Conservation of momentum:

$$(M+m)\,\dot{x}_0 = m\,v \quad \text{or} \quad \dot{x}_0 = \dot{x}(t{=}0) = \frac{m\,v}{M+m}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{4\,k}{M+m}}$$

Complete solution:

$$x(t) = A_0 \sin(\omega_n\, t + \phi_0)$$

where

$$A_0 = \left\{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right\}^{\frac{1}{2}} = \left\{\frac{m^2\,g^2}{16\,k^2} + \frac{m^2\,v^2}{(M+m)\,4\,k}\right\}^{\frac{1}{2}}$$

and

$$\phi_0 = \tan^{-1}\left(\frac{x_0\,\omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(-\frac{m\,g}{4\,k}\sqrt{\frac{4\,k}{(M+m)}}\,\frac{(M+m)}{m\,v}\right) = \tan^{-1}\left(-\frac{g\sqrt{M+m}}{v\sqrt{4\,k}}\right)$$

Since v = 600, m = 12/386.4, M = 100/386.4, k = 100, we find

$$A_0 = \left\{ \left(\frac{12\ (386.4)}{4\ (100)\ (386.4)} \right)^2 + \left(\frac{12\ (600)}{386.4} \right)^2 \frac{386.4}{112\ (400)} \right\}^{\frac{1}{2}} = 1.7308 \text{ in}$$

$$\phi_0 = \tan^{-1} \left(- \frac{386.4 \sqrt{112}}{\sqrt{386.4\ (600)} \sqrt{400}} \right) = \tan^{-1} (-0.01734) = -0.9934 \text{ deg}$$

(b) x = 0 at static equilibrium position: $x_0 = x(t{=}0) = 0$. Conservation of momentum gives:

$$M\, \dot{x}_0 = m\, v \quad \text{or} \quad \dot{x}_0 = \dot{x}(t{=}0) = \frac{m\, v}{M}$$

Complete solution:

$$x(t) = A_0 \sin (\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} = \left\{ \frac{m^2\, v^2\, (M)}{M^2\, 4\, k} \right\}^{\frac{1}{2}} = \frac{m\, v}{\sqrt{4\, k\, M}} = \frac{12\ (600) \sqrt{386.4}}{386.4 \sqrt{4\ (100)\ (100)}} = 1.8314 \text{ in}$$

$$\phi_0 = \tan^{-1} \left(\frac{x_0\, \omega_n}{\dot{x}_0} \right) = \tan^{-1} (0) = 0$$

2.17 $k_1 = \dfrac{3 E_1 I_1}{l_1^3}$ (at tip) ; $k_2 = \dfrac{48 E_2 I_2}{l_2^3}$ (at middle)

$k_{eq} = k_1 + k_2$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\left(\frac{3 E_1 I_1}{l_1^3} + \frac{48 E_2 I_2}{l_2^3} \right) \frac{g}{W}}$$

2.18 $k = \dfrac{AE}{l} = \dfrac{\left\{ \frac{\pi}{4} (0.01)^2 \right\} \left\{ 2.07 \times 10^{11} \right\}}{20} = 0.8129 \times 10^6$ N/m

m = 1000 kg

$$\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{0.8129 \times 10^6}{1000} \right)^{1/2} = 28.5114 \text{ rad/sec}$$

$\dot{x}_0 = 2$ m/s, $x_0 = 0$ (suddenly stopped while it has velocity)

Period of ensuing vibration $= \tau_n = \dfrac{2\pi}{\omega_n} = \dfrac{2\pi}{28.5114} = 0.2204$ sec

Amplitude $= A = \dot{x}_0 / \omega_n = 2/28.5114 = 0.07015$ m

---

2.19 $\omega_n = 2$ Hz $= 12.5664$ rad/sec $= \sqrt{\dfrac{k}{m}}$

$\sqrt{k} = 12.5664 \sqrt{m}$

$$\omega_n' = \sqrt{\frac{k'}{m'}} = \sqrt{\frac{k}{m+1}} = 6.2832 \text{ rad/sec}$$

$$\sqrt{k} = 6.2832\sqrt{m+1}$$
$$= 12.5664\sqrt{m}$$

$$\sqrt{m+1} = 2\sqrt{m}\ , \quad m = \frac{1}{3}\ \text{kg}$$

$$k = (12.5664)^2\ m = 52.6381\ \text{N/m}$$

(2.20) Cable stiffness $= k = \dfrac{A\,E}{\ell} = \dfrac{1}{4}\left[\dfrac{\pi}{4}\,(0.01)^2\right] 2.07\,(10^{11}) = 4.0644\,(10^6)$ N/m

$$\tau_n = 0.1 = \frac{1}{f_n} = \frac{2\pi}{\omega_n}$$

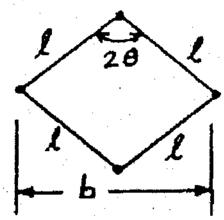
$$\omega_n = \frac{2\pi}{0.1} = 20\,\pi = \sqrt{\frac{k}{m}}$$

Hence

$$m = \frac{k}{\omega_n^2} = \frac{4.0644\,(10^6)}{(20\,\pi)^2} = 1029.53\ \text{kg}$$

(2.21) $b = 2\ell\sin\theta$

Neglect masses of links.



(a) $k_{eq} = k\left(\dfrac{4\ell^2 - b^2}{b^2}\right) = k\left(\dfrac{4\ell^2 - 4\ell^2\sin^2\theta}{4\ell^2\sin^2\theta}\right)$

$$= k\left(\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$\omega_n = \sqrt{\dfrac{k_{eq}}{m}} = \sqrt{\dfrac{k\,g\,\operatorname{cosec}^2\theta}{W}}$ (from solution of problem 1.8)

(b) $\omega_n = \sqrt{\dfrac{k\,g}{W}}$ since $k_{eq} = k$.

(2.22) $y = \sqrt{\ell^2 - (\ell\sin\theta - x)^2} - \ell\cos\theta = \sqrt{\ell^2\cos^2\theta - x^2 + 2\,\ell\,x\sin\theta} - \ell\cos\theta$

$$= \ell\cos\theta\sqrt{1 - \frac{x^2}{\ell^2\cos^2\theta} + \frac{2\,\ell\,x\sin\theta}{\ell^2\cos^2\theta}} - \ell\cos\theta$$

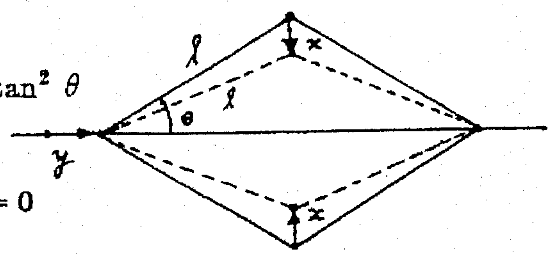
$$\frac{1}{2}\,k_{eq}\,x^2 = \frac{1}{2}\,k_1\,y^2 + \frac{1}{2}\,k_2\,y^2$$

where

$$y \approx \ell\cos\theta\left\{1 - \frac{1}{2}\,\frac{x^2}{\ell^2\cos^2\theta} + \frac{1}{2}\,\frac{2\,\ell\,x\sin\theta}{\ell^2\cos^2\theta}\right\} - \ell\cos\theta$$

$$\approx \frac{x\sin\theta}{\cos\theta} = x\tan\theta$$

Thus k_{eq} can be expressed as



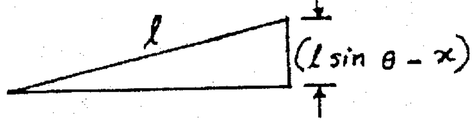
$$k_{eq} = (k_1 + k_2) \tan^2 \theta$$

Equation of motion:

$$m \ddot{x} + k_{eq} x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{(k_1 + k_2)\, g}{W}} \tan \theta$$



2.23 (a) Neglect masses of rigid links. Let x = displacement of W. Springs are in series.

$$k_{eq} = \frac{k}{2}$$

Equation of motion:

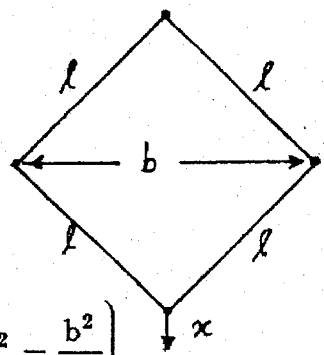
$$m \ddot{x} + k_{eq} x = 0$$

Natual frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{2\,m}}$$

(b) Under a displacement of x of mass, each spring will be compressed by an an amount:

$$x_s = x \frac{2}{b} \sqrt{\ell^2 - \frac{b^2}{4}}$$



Equivalent spring constant:

$$\frac{1}{2} k_{eq} x^2 = 2 \left(\frac{1}{2} k x_s^2 \right)$$

$$\text{or} \quad k_{eq} = 2 k \left(\frac{x_s}{x} \right)^2 = 2 k \left(\frac{4}{b^2} \right) \left(\ell^2 - \frac{b^2}{4} \right) = \frac{8 k}{b^2} \left(\ell^2 - \frac{b^2}{4} \right)$$

Equation of motion:

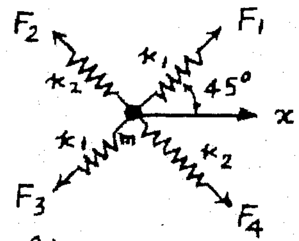
$$m \ddot{x} + k_{eq} x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{8 k}{b^2\, m} \left(\ell^2 - \frac{b^2}{4} \right)}$$

(2.24) $F_1 = F_3 = k_1\, x \cos 45°$

$F_2 = F_4 = k_2\, x \cos 135°$



$$F = \text{force along } x = F_1 \cos 45° + F_2 \cos 135° + F_3 \cos 45° + F_4 \cos 135°$$

$$= 2x\left(k_1 \cos^2 45° + k_2 \cos^2 135°\right)$$

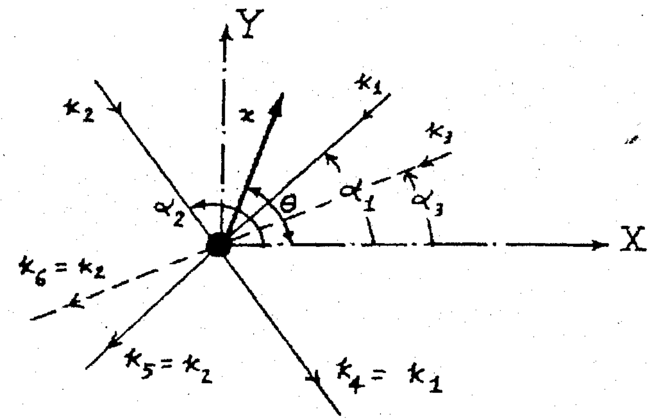
$$k_{eq} = \frac{F}{x} = 2\left(\frac{k_1}{2} + \frac{k_2}{2}\right) = k_1 + k_2$$

Equation of motion: $m\ddot{x} + (k_1 + k_2)x = 0$

(2.25) Let α_i denote the angle made by i^{th} spring with respect to X axis.

Let x = displacement of mass along the direction defined by θ.

If k_{eq} = equivalent spring constant, the equivalence of potential energies gives



$$\frac{1}{2} k_{eq}\, x^2 = \frac{1}{2} \sum_{i=1}^{6} k_i \left\{x \cos(\theta - \alpha_i)\right\}^2$$

$$k_{eq} = \sum_{i=1}^{6} k_i \cos^2(\theta - \alpha_i) = \sum_{i=1}^{6} k_i \left(\cos\theta \cos\alpha_i + \sin\theta \sin\alpha_i\right)^2$$

$$= \sum_{i=1}^{6} k_i \left(\cos^2\alpha_i \cos^2\theta + \sin^2\alpha_i \sin^2\theta\right)$$

$$+ 2 \sum_{i=1}^{6} \left(\cos\alpha_i \sin\alpha_i \cos\theta \sin\theta\right)$$

Natural frequency $= \omega_n = \sqrt{\dfrac{k_{eq}}{m}}$

For ω_n to be independent of θ, $\quad \displaystyle\sum_{i=1}^{6} k_i \cos^2\alpha_i = \sum_{i=1}^{6} k_i \sin^2\alpha_i \quad \cdots (E_1)$

and $\quad \displaystyle\sum_{i=1}^{6} k_i \cos\alpha_i \sin\alpha_i = 0 \quad \cdots (E_2)$

(E_1) and (E_2) can be rewritten as

$$\sum_{i=1}^{6} k_i \left(\frac{1}{2} + \frac{1}{2}\cos 2\alpha_i\right) = \sum_{i=1}^{6} k_i \left(\frac{1}{2} - \frac{1}{2}\cos 2\alpha_i\right)$$

and $\quad \displaystyle\frac{1}{2} \sum_{i=1}^{6} k_i \sin 2\alpha_i = 0$

i.e. $\quad \displaystyle\sum_{i=1}^{6} k_i \cos 2\alpha_i = 0 \quad --- (E_3)$

and $\quad \displaystyle\sum_{i=1}^{6} k_i \sin 2\alpha_i = 0 \quad --- (E_4)$

In the present example, (E_3) and (E_4) become

$$k_1 \cos 60^\circ + k_2 \cos 240^\circ + k_3 \cos 2\alpha_3 + k_1 \cos 420^\circ + k_2 \cos 600^\circ + k_3 \cos(360^\circ + 2\alpha_3) = 0$$

$$k_1 \sin 60^\circ + k_2 \sin 240^\circ + k_3 \sin 2\alpha_3 + k_1 \sin 420^\circ + k_2 \sin 600^\circ + k_3 \sin(360^\circ + 2\alpha_3) = 0$$

i.e., $\left.\begin{array}{l} k_1 - k_2 + 2 k_3 \cos 2\alpha_3 = 0 \\ \sqrt{3}\, k_1 - \sqrt{3}\, k_2 + 2 k_3 \sin 2\alpha_3 = 0 \end{array}\right\}$; $\begin{array}{l} 2 k_3 \cos 2\alpha_3 = k_2 - k_1 \ldots (E_5) \\ 2 k_3 \sin 2\alpha_3 = \sqrt{3}\,(k_2 - k_1) \ldots (E_6) \end{array}$

Squaring (E_5) and (E_6) and adding,

$$4 k_3^2 = (k_2 - k_1)^2 (1 + 3)$$

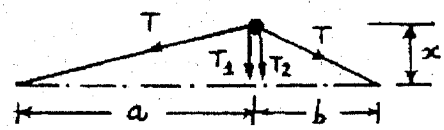
$$\therefore k_3 = \pm (k_2 - k_1) \Rightarrow k_3 = |k_2 - k_1|$$

Dividing (E_6) by (E_5),

$$\tan 2\alpha_3 = \sqrt{3}$$

$$\therefore \alpha_3 = \tfrac{1}{2} \tan^{-1}(\sqrt{3}) = 30^\circ$$

2.26 $T_1 = \dfrac{x}{a} T, \qquad T_2 = \dfrac{x}{b} T$



(a) $m\ddot{x} + (T_1 + T_2) = 0$

$$m\ddot{x} + \left(\frac{T}{a} + \frac{T}{b}\right) x = 0$$

(b) $\omega_n = \sqrt{\dfrac{\frac{T}{a} + \frac{T}{b}}{m}} = \sqrt{\dfrac{T}{m\,a\,b}(a+b)}$

2.27 $m = \dfrac{160}{386.4} \dfrac{\text{lb—sec}^2}{\text{inch}}$, $k = 10$ lb/inch.

Velocity of jumper as he falls through 200 ft:

$$m\,g\,h = \frac{1}{2} m\,v^2 \quad \text{or} \quad v = \sqrt{2\,g\,h} = \sqrt{2\,(386.4)\,(200\,(12))} = 1{,}361.8811 \text{ in/sec}$$

About static equilibrium position:

$$x_0 = x(t{=}0) = 0, \quad \dot{x}_0 = \dot{x}(t{=}0) = 1{,}361.8811 \text{ in/sec}$$

Response of jumper: $\qquad x(t) = A_0 \sin(\omega_n t + \phi_0)$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2 \right\}^{\frac{1}{2}} = \frac{\dot{x}_0}{\omega_n} = \frac{\dot{x}_0 \sqrt{m}}{\sqrt{k}} = \frac{1361.8811}{\sqrt{10}} \sqrt{\left(\frac{160}{386.4}\right)} = 277.1281 \text{ in}$$

and $\qquad \phi_0 = \tan^{-1}\left(\dfrac{x_0\,\omega_n}{\dot{x}_0}\right) = 0$

2.28 The natural frequency of a vibrating rope is given by (see Problem 2.26):

$$\omega_n = \sqrt{\frac{T\,(a+b)}{m\,a\,b}}$$

where T = tension in rope, m = mass, and a and b are lengths of the rope on both sides of the mass. For the given data:

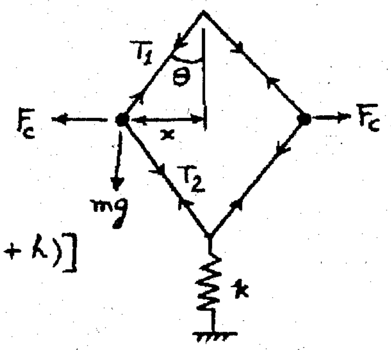
$$10 = \left\{ \frac{T\,(80+160)}{\left(\frac{120}{386.4}\right)(80)\,(160)} \right\}^{\frac{1}{2}} = \sqrt{T\,(0.060375)}$$

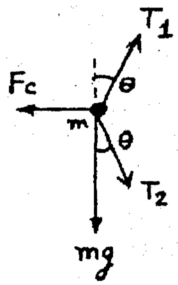
which yields

$$T = \frac{100}{0.060375} = 1{,}656.3147 \text{ lb}$$

2.29 When $\omega = 0$, total vertical height $= 2l + h$

When $\omega \neq 0$, total vertical height $= (2l\cos\theta + h)$





$F_c = m\,x\,\omega^2$

$x = l\sin\theta$

$$\text{spring force} = k\left[2l + h - (2l\cos\theta + h)\right] = 2kl\,(1-\cos\theta)$$

For vertical equilibrium of mass m,

$$mg + T_2\cos\theta = T_1\cos\theta \quad \text{--- } (E_1)$$

For horizontal equilibrium, $\quad F_c = (T_1 + T_2)\sin\theta$

$$T_2 = (F_c - T_1\sin\theta)/\sin\theta \quad \text{--- } (E_2)$$

From (E_2), (E_1) can be expressed as

$$mg + \left(\frac{F_c - T_1\sin\theta}{\sin\theta}\right)\cos\theta = T_1\cos\theta$$

i.e. $$T_1 = \frac{mg + F_c\cot\theta}{2\cos\theta} = \frac{mg + m\omega^2 l\cos\theta}{2\cos\theta}$$

$$T_2 = \frac{F_c - T_1\sin\theta}{\sin\theta} = \frac{m\,x\,\omega^2 - \frac{mg}{2}\tan\theta - \frac{m\omega^2 l}{2}\sin\theta}{\sin\theta}$$

$$= \frac{m\,l\,\omega^2}{2} - \frac{mg}{2\cos\theta}$$

$$\text{spring force} = 2kl\,(1-\cos\theta) = 2T_2\cos\theta$$

$$= m\,l\,\omega^2\cos\theta - mg$$

$$\cos\theta = \left(\frac{2kl + mg}{2kl + m\,l\,\omega^2}\right) \quad \text{--- } (E_3)$$

This equation defines the equilibrium position of mass m.
For small oscillations about the equilibrium position,
Newton's second law gives

$$2m\ddot{y} + k y = 0 \quad , \qquad \omega_n = \sqrt{\frac{2k}{m}}$$

(2.30) (a) Let P = total spring force, F = centrifugal force acting on each ball. Equilibrium of moments about the pivot of bell crank lever (O) gives:

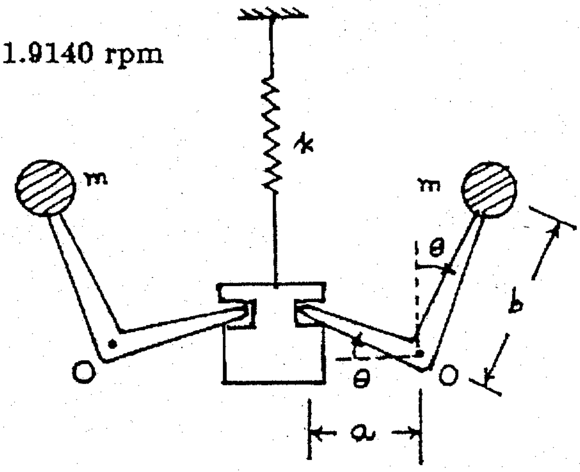
$$F\left(\frac{20}{100}\right) = \frac{P}{2}\left(\frac{12}{100}\right) \tag{1}$$

When $P = 10^4\left(\frac{1}{100}\right) = 100$ N, and

$$F = m\,r\,\omega^2 = m\,r\left(\frac{2\,\pi\,N}{60}\right)^2 = \frac{25}{9.81}\left(\frac{16}{100}\right)\left(\frac{2\,\pi\,N}{60}\right)^2 = 0.004471\ N^2$$

where N = speed of the governor in rpm. Equation (1) gives:

$$0.004471\ N^2\ (0.2) = \frac{100}{2}\ (0.12) \quad \text{or} \quad N = 81.9140 \text{ rpm}$$



(b) Consider a small displacement of the ball arm about the vertical position. Equilibrium about point O gives:

$$(m\,b^2)\,\ddot{\theta} + (k\,a\sin\theta)\,a\cos\theta = 0 \tag{2}$$

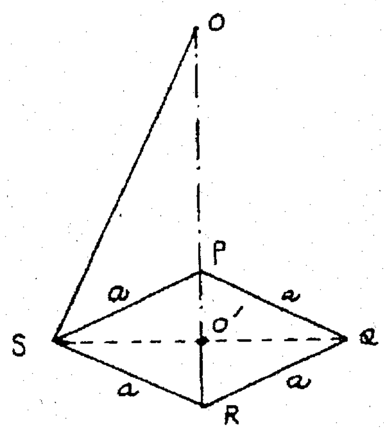
For small vallues of θ, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, and hence Eq. (2) gives

$$m\,b^2\,\ddot{\theta} + k\,a^2\,\theta = 0$$

from which the natural frequency can be determined as

$$\omega_n = \left\{\frac{k\,a^2}{m\,b^2}\right\}^{\frac{1}{2}} = \left\{(10)^4\left(\frac{0.12}{0.20}\right)^2\frac{9.81}{25}\right\}^{\frac{1}{2}} = 37.5851 \text{ rad/sec}$$

2.31 $SO' = \frac{a}{\sqrt{2}}$, $OO' = h$, $OS = \sqrt{h^2 + \frac{a^2}{2}}$



When each wire stretches by x_s, let the resulting vertical displacement of the platform be x.

$$OS + x_s = \sqrt{(h+x)^2 + \frac{a^2}{2}}$$

$$x_s = \sqrt{h^2 + \frac{a^2}{2}} \left\{ \sqrt{\frac{(h+x)^2 + \frac{a^2}{2}}{h^2 + \frac{a^2}{2}}} - 1 \right\}$$

$$= \sqrt{h^2 + \frac{a^2}{2}} \left[\sqrt{1 + \left\{ \frac{2hx + x^2}{\left(h^2 + \frac{a^2}{2}\right)} \right\}} - 1 \right]$$

For small x, x^2 is negligible compared to $2hx$ and $\sqrt{1+\theta} \simeq 1 + \frac{\theta}{2}$ and hence

$$x_s = \sqrt{h^2 + \frac{a^2}{2}} \left[1 + \frac{hx}{\left(h^2 + \frac{a^2}{2}\right)} - 1 \right] = \frac{h}{\sqrt{h^2 + \frac{a^2}{2}}} \cdot x$$

Potential energy equivalence gives

$$\frac{1}{2} k_{eq} x^2 = 4 \left(\frac{1}{2} k \; x_s^2 \right)$$

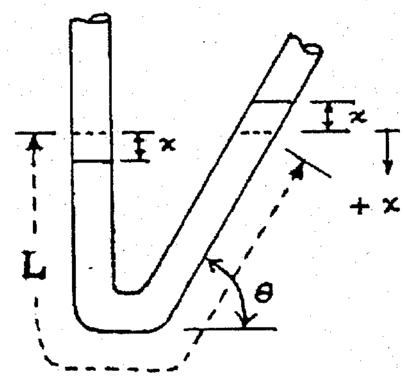
$$k_{eq} = 4k \left(\frac{x_s}{x} \right)^2 = \frac{4kh^2}{\left(h^2 + \frac{a^2}{2}\right)}$$

Equation of motion of M:

$$M \ddot{x} + k_{eq} x = 0$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{(k_{eq}/M)^{1/2}} = \frac{\pi \sqrt{M}}{h} \left(\frac{2h^2 + a^2}{2k} \right)^{1/2}$$

2.32 Equation of motion:



$$m \ddot{x} = \sum F_x$$

i.e., $(L A \rho) \ddot{x} = -2 (A x \rho g)$

i.e., $\ddot{x} + \frac{2g}{L} x = 0$

where A = cross-sectional area of the tube and ρ = density of mercury. Thus the natural frequency is given by:

$$\omega_n = \sqrt{\frac{2g}{L}}$$

(2.33) Assume same area of cross section for all segments of the cable. Speed of blades = 300 rpm = 5 Hz = 31.416 rad/sec.

$$\omega_n^2 = \frac{k_{eq}}{m} = (2\ (31.416))^2 = (62.832)^2$$

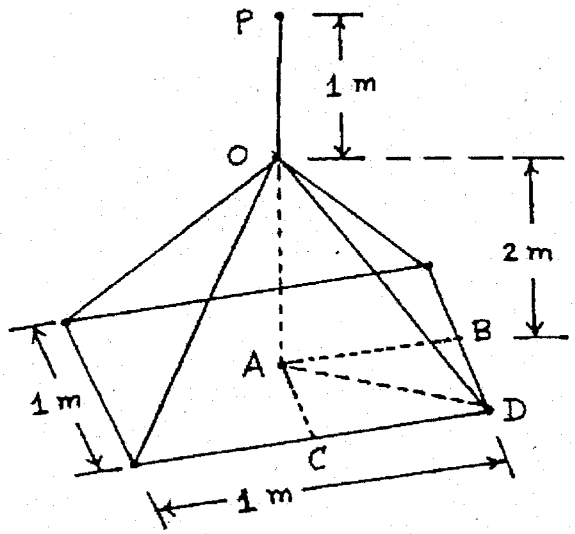
$$k_{eq} = m\ \omega_n^2 = 250\ (62.832)^2 = 98.6965\ (10^4)\ \text{N/m} \tag{1}$$

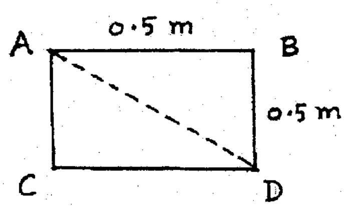
$$AD = \sqrt{0.5^2 + 0.5^2} = 0.7071\ \text{m}\ ,\quad OD = \sqrt{2^2 + 0.7071^2} = 2.1213\ \text{m}$$

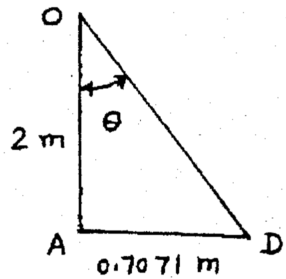
Stiffness of cable segments:

$$k_{PO} = \frac{A\ E}{\ell_{PO}} = \frac{A\ (207)\ (10^9)}{1} = 207\ (10^9)\ A\ \text{N/m}$$

$$K_{OD} = \frac{A\ E}{\ell_{OD}} = \frac{A\ (207)\ (10^9)}{2.1213} = 97.5817\ (10^9)\ A\ \text{N/m}$$







$$\theta = \tan^{-1}\left(\frac{0.7071}{2}\right)$$
$$= 19.4710^\circ$$

The total sttiffness of the four inclined cables (k_{ic}) is given by:

$$k_{ic} = 4\ k_{OD}\ \cos^2\theta$$
$$= 4\ (97.5817)\ (10^9)\ A\ \cos^2 19.4710^\circ = 346.9581\ (10^9)\ A\ \text{N/m}$$

Equivalent stiffness of vertical and inclined cables is given by:

$$\frac{1}{k_{eq}} = \frac{1}{k_{PO}} + \frac{1}{k_{ic}}$$

i.e., $$k_{eq} = \frac{k_{PO}\ k_{ic}}{k_{PO} + k_{ic}}$$

$$= \frac{(207\ (10^9)\ A)\ (346.9581\ (10^9)\ A)}{(207\ (10^9)\ A) + (346.9581\ (10^9)\ A)} = 129.6494\ (10^9)\ A\ \text{N/m} \tag{2}$$

Equating k_{eq} given by Eqs. (1) and (2), we obtain the area of cross section of cables as:

$$A = \frac{98.6965\ (10^4)}{129.6494\ (10^9)} = 7.6126\ (10^{-6})\ \text{m}^2$$

(2.34)

$$\frac{1}{2\,\pi}\left\{\frac{k_1}{m}\right\}^{\frac{1}{2}} = 5\ ;\ \frac{k_1}{m} = 4\,(\pi)^2\,(25) = 986.9651$$

$$\frac{1}{2\,\pi}\left\{\frac{k_1}{m+5000}\right\}^{\frac{1}{2}} = 4.0825\ ;\ \frac{k_1}{m+5000} = 4\,(\pi)^2\,(16.6668) = 657.9822$$

Using $k_1 = \frac{A\,E}{\ell_1}$ we obtain

$$\frac{k_1}{m} = \frac{A\,E}{\ell_1\,m} = \frac{A\,(207)\,(10^9)}{2\,m} = 986.9651$$

$$\text{i.e.,}\quad A = 9.5359\,(10^{-9})\,m \tag{1}$$

Also

$$\frac{k_1}{m+5000} = \frac{A\,E}{\ell_1\,(m+5000)} = 657.9822$$

$$\text{i.e.,}\quad \frac{A}{m+5000} = 6.3573\,(10^{-9}) \tag{2}$$

Using Eqs. (1) and (2), we obtain

$$A = 9.5359\,(10^{-9})\,m = 6.3573\,(10^{-9})\,m + 31.7865\,(10^{-6})$$

$$\text{i.e.,}\quad 3.1786\,(10^{-9})\,m = 31.7865\,(10^{-6})$$

$$\text{i.e.,}\quad m = 10000.1573\ \text{kg} \tag{3}$$

Equations (1) and (3) yield

$$A = 9.5359\,(10^{-9})\,m = 9.5359\,(10^{-9})\,(10000.1573) = 0.9536\,(10^{-4})\ m^2$$

(2.35) Longitudinal Vibration:

Let W_1 = part of weight W carried by length a of shaft

$W_2 = W - W_1$ = weight carried by length b

x = Elongation of length $a = \frac{W_1\,a}{A\,E}$

y = shortening of length $b = \frac{(W - W_1)(\ell - a)}{AE}$

E = Young's modulus

A = area of cross-section $= \pi d^2/4$

Since $x = y$, $\quad W_1 = \frac{W(\ell - a)}{\ell}$

x = elongation or static deflection of length $a = \frac{W a\,(\ell - a)}{AE\,\ell}$

Considering the shaft of length a with end mass W_1/g as a spring-mass system,

$$\omega_n = \sqrt{\frac{g}{x}} = \left(\frac{g\,\ell\,AE}{W a\,(\ell - a)}\right)^{1/2}$$

Transverse vibration:

spring constant of a fixed-fixed beam with off-center load

$$= k = \frac{3 E I \ell^3}{a^3 b^3} = \frac{3 E I \ell^3}{a^3 (\ell - a)^3}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \left\{ \frac{3 E I \ell^3 g}{W a^3 (\ell - a)^3} \right\}^{1/2} \quad \text{with} \quad I = (\pi d^4 / 64) = \text{moment of inertia}$$

Torsional vibration:

If flywheel is given an angular deflection θ, resisting torques offered by lengths a and b are $\frac{GJ\theta}{a}$ and $\frac{GJ\theta}{b}$.

Total resisting torque $= M_t = GJ\left(\frac{1}{a} + \frac{1}{b}\right)\theta$

$$k_t = \frac{M_t}{\theta} = GJ\left(\frac{1}{a} + \frac{1}{b}\right) \quad \text{where} \quad J = \frac{\pi d^4}{32} = \text{polar moment of inertia}$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left\{ \frac{GJ}{J_0} \left(\frac{1}{a} + \frac{1}{b}\right) \right\}^{1/2}$$

where J_0 = mass polar moment of inertia of the flywheel.

2.36 $m_{eq_{end}}$ = equivalent mass of a uniform beam at the free end (see Problem 2.38) =

$$\frac{33}{140}\, m = \frac{33}{140} \left\{ 1\ (1)\ (150 \times 12)\ \frac{0.283}{386.4} \right\} = 0.3107$$

Stiffness of tower (beam) at free end:

$$k_b = \frac{3 E I}{L^3} = \frac{3\ (30 \times 10^6)\ (\frac{1}{12}\ (1)\ (1^3))}{(150 \times 12)^3} = 0.001286 \text{ lb/in}$$

Length of each cable:

$$OA = \sqrt{2} = 1.4142 \text{ ft}, \quad OB = \sqrt{2}\ 15 = 21.2132 \text{ ft}, \quad AB = OB - OA = 19.7990 \text{ ft}$$

$$TB = \sqrt{TA^2 + AB^2} = \sqrt{100^2 + 19.7990^2} = 101.9412 \text{ ft}$$

$$\tan\theta = \frac{AT}{AB} = \frac{100}{19.7990} = 5.0508, \quad \theta = 78.8008^\circ$$

Axial stiffness of each cable:

$$k = \frac{A E}{\ell} = \frac{(0.5)\ (30 \times 10^6)}{(101.9412 \times 12)} = 12261.971 \text{ lb/in}$$

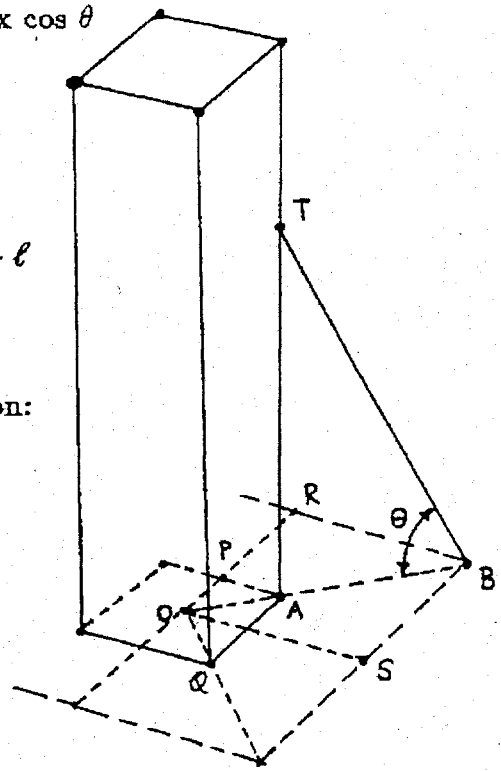
Axial extension of each cable (y_c) due to a horizontal displacement of x of tower:

$$\ell_1^2 = \ell^2 + x^2 - 2\,\ell\, x \cos(180^\circ - \theta) = \ell^2 + x^2 + 2\,\ell\, x \cos\theta$$

$$\text{or} \quad \ell_1 = \ell \left\{ 1 + \left(\frac{x}{\ell} \right)^2 + 2\, \frac{x}{\ell} \cos\theta \right\}^{\frac{1}{2}}$$

$$y_c = \ell_1 - \ell \approx \ell \left(1 + \frac{1}{2}\, \frac{x^2}{\ell^2} + \frac{1}{2}\,(2)\, \frac{x}{\ell} \cos\theta \right) - \ell$$

$$= \ell + x \cos\theta - \ell = x \cos\theta$$



Equivalent stiffness of each cable in horizontal direction:

$$\frac{1}{2}\, k\, y_c^2 = \frac{1}{2}\, k_{eq_c}\, x^2 \quad \text{or} \quad k_{eq_c} = k \left(\frac{y_c}{x} \right)^2 = k \cos^2\theta$$

This gives

$$k_{eq_c} = (12261.971) \cos^2 78.8008^\circ = 462.5419 \text{ lb/in}$$

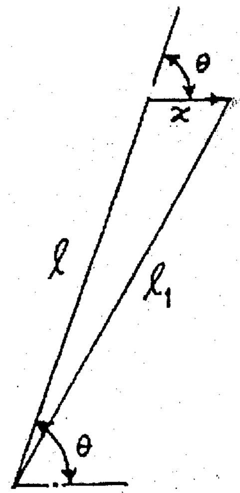
In order to use the relation

$$k_{eq_{end}} = k_b + 4\, k_{eq_c} \left(\frac{y_{L1}}{y_L} \right)^2$$

we find

$$\frac{y_{L1}}{y_L} = \left(\frac{F\, L_1^2\, (3\,L - L_1)}{6\,E\,I}\; \frac{3\,E\,I}{F\,L^3} \right) = \frac{L_1^2\,(3\,L - L_1)}{2\,L^3}$$

$$= \frac{100^2\,(3\,(150) - 100)}{2\,(150)^3} = 0.5185$$

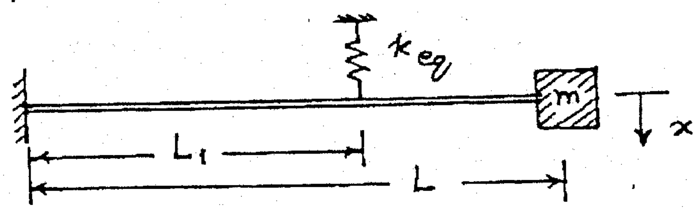


Thus

$$k_{eq_{end}} = k_b + 4\, k_{eq_c}\,(0.5185)^2 = 0.001286 + 4\,(462.5419)\,(0.5185)^2$$

$$= 497.4045 \text{ lb/in}$$

Natural frequency:



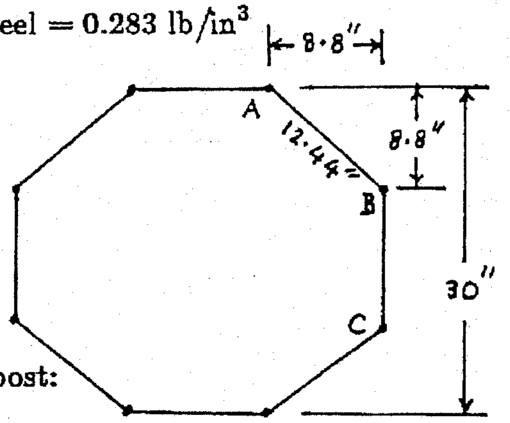
$$\omega_n = \left\{ \frac{k_{eq_{end}}}{m_{eq_{end}}} \right\}^{\frac{1}{2}} = \left\{ \frac{497.4045}{0.3107} \right\}^{\frac{1}{2}} = 40.0114 \text{ rad/sec}$$

2.37 Sides of the sign:

$$AB = \sqrt{8.8^2 + 8.8^2} = 12.44 \text{ in} \;\; ; \;\; BC = 30 - 8.8 - 8.8 = 12.4 \text{ in}$$

$$\text{Area} = 30\,(30) - 4\,(\frac{1}{2}\,(8.8)\,(8.8)) = 745.12 \text{ in}^2$$

$$\text{Thickness} = \frac{1}{8} \text{ in} \;\; ; \;\; \text{Weight density of steel} = 0.283 \text{ lb/in}^3$$



$$\text{Weight of sign} = (0.283)(\frac{1}{8})(745.12) = 26.64 \text{ lb}$$

$$\text{Weight of sign post} = (72)\,(2)\,(\frac{1}{4})\,(0.283) = 10.19 \text{ lb}$$

Stiffness of sign post (cantilever beam):

$$k = \frac{3\,E\,I}{\ell^3}$$

Area moments of inertia of the cross section of the sign post:

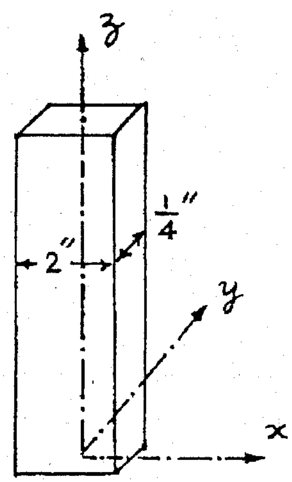
$$I_{xx} = \frac{1}{12}\,(2)\,(\frac{1}{4})^3 = \frac{1}{384} \text{ in}^4$$

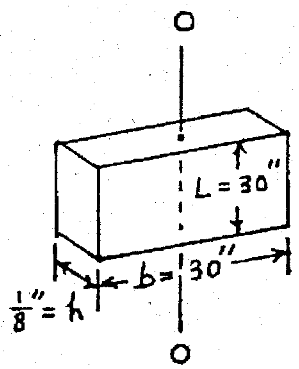
$$I_{yy} = \frac{1}{12}\,(\frac{1}{4})\,(2)^3 = \frac{1}{6} \text{ in}^4$$

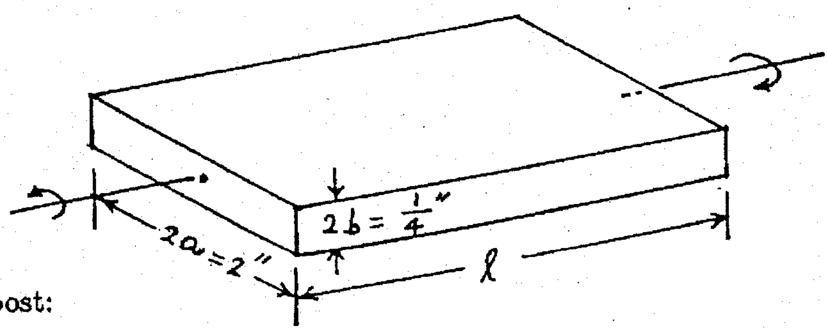
Bending stiffnesses of the sign post:

$$k_{xz} = \frac{3\,E\,I_{yy}}{\ell^3} = \frac{3\,(30\,(10^6))(\frac{1}{6})}{72^3} = 40.1877 \text{ lb/in}$$

$$k_{yz} = \frac{3\,E\,I_{xx}}{\ell^3} = \frac{3\,(30\,(10^6))\,(\frac{1}{384})}{72^3} = 0.6279 \text{ lb/in}$$







Torsional stiffness of the sign post:

$$k_t = 5.33 \frac{a\, b^3}{\ell} G \left\{ 1 - 0.63 \frac{b}{a} \left(1 - \frac{b^4}{12\, a^4} \right) \right\}$$

(See Ref: N. H. Cook, *Mechanics of Materials for Design*, McGraw-Hill, New York, 1984, p. 342).

Thus

$$k_t = 5.33 \left\{ \frac{(1)\, (\frac{1}{8})^3}{72} \right\} (11.5\, (10^6)) \left\{ 1 - (0.63)\, (\frac{1}{8}) \left(1 - \frac{(\frac{1}{8})^4}{12\, (1)^4} \right) \right\}$$

$$= 1531.7938 \text{ lb—in/rad}$$

Natural frequency for bending in xz plane:

$$\omega_{xz} = \left\{ \frac{k_{xz}}{m} \right\}^{\frac{1}{2}} = \left\{ \frac{40.1877}{\left(\frac{26.64}{386.4} \right)} \right\}^{\frac{1}{2}} = 24.1434 \text{ rad/sec}$$

Natural frequency for bending in yz plane:

$$\omega_{yz} = \left\{ \frac{k_{yz}}{m} \right\} = \left\{ \frac{0.6279}{\left(\frac{26.64}{386.4} \right)} \right\}^{\frac{1}{2}} = 3.0178 \text{ rad/sec}$$

By approximating the shape of the sign as a square of side 30 in (instead of an octagon), we can find its mass moment of inertia as:

$$I_{oo} = \frac{\gamma L}{3} (b^3\, h + h^3\, b) = \left(\frac{0.283}{386.4} \right) \left(\frac{30}{3} \right) \left(30^3\, (\frac{1}{8}) + (\frac{1}{8})^3\, (30) \right) = 24.7189$$

Natural torsional frequency:

$$\omega_t = \left\{ \frac{k_t}{I_{oo}} \right\}^{\frac{1}{2}} = \left\{ \frac{1531.7938}{24.7189} \right\}^{\frac{1}{2}} = 7.8720 \text{ rad/sec}$$

Thus the mode of vibration (close to resonance) is torsion in xy plane.

2.38 (a) Pivoted: Let $l = h$.

$k_{eq} = 4\ k_{column} = 4 \left(\frac{3EI}{l^3} \right) = \frac{12EI}{l^3}$

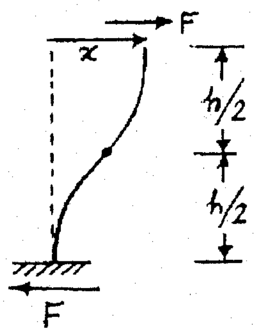
Let $m_{eff\,1}$ = effective mass due to self weight of columns

Equation of motion: $\left(\frac{W}{g} + m_{eff\,1} \right) \ddot{x} + k_{eq}\, x = 0$

Natural frequency of horizontal vibration = $\omega_n = \sqrt{\dfrac{12EI}{l^3 \left(\frac{W}{g} + m_{eff\,1} \right)}}$

(b) Fixed:

since the joint between column and floor does not permit rotation, each column will bend with inflection point at middle.



When force F is applied at ends,

$$x = 2 \frac{F(l/2)^3}{3EI} = \frac{Fl^3}{12EI}$$

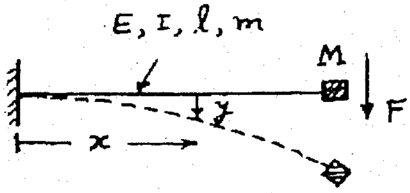
$$k_{column} = \frac{12EI}{l^3} \quad ; \quad k_{eq} = 4\,k_{column} = \frac{48EI}{l^3}$$

Let m_{eff2} = effective mass of each column at top end

Equation of motion: $\left(\frac{W}{g} + m_{eff2}\right)\ddot{x} + k_{eq}\,x = 0$

Natural frequency of horizontal vibration $= \omega_n = \sqrt{\dfrac{48EI}{l^3\left(\frac{W}{g} + m_{eff2}\right)}}$

Effective mass (due to self weight):



(a) Let m_{eff1} = effective part of mass of beam (m) at end.

Thus vibrating inertia force at end is due to $(M + m_{eff1})$.

Assume deflection shape during vibration same as the static deflection shape with a tip load:

$$y(x,t) = Y(x)\cos(\omega_n t - \phi) \quad \text{where} \quad Y(x) = \frac{F x^2 (3l - x)}{6EI}.$$

$$Y(x) = \frac{Y_0}{2l^3}\,x^2(3l - x) \quad \text{where} \quad Y_0 = \frac{Fl^3}{3EI} = \text{max. tip deflection}$$

$$y(x,t) = \frac{Y_0}{2l^3}(3x^2 l - x^3)\cos(\omega_n t - \phi) \qquad (E_1)$$

Max. strain energy of beam = Max. work by force F

$$= \frac{1}{2} F Y_0 = \frac{3}{2}\frac{EI}{l^3} Y_0^2 \qquad (E_2)$$

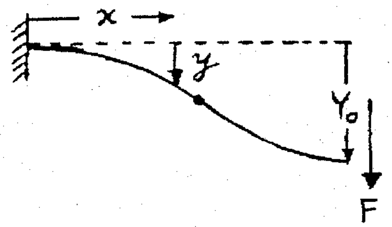
Max. kinetic energy due to distributed mass of beam

$$= \frac{1}{2}\frac{m}{l}\int_0^l \dot{y}^2(x,t)\Big|_{max} dx + \frac{1}{2}\left(\dot{y}_{max}\right)^2 M$$

$$= \frac{1}{2}\omega_n^2 Y_0^2\left(\frac{33}{140}m\right) + \frac{1}{2}\omega_n^2 Y_0^2 M \qquad (E_3)$$

$$\therefore \quad m_{eff\,1} = \frac{33}{140}m = 0.2357\,m$$

(b) Let $Y(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$



$$Y(0) = 0, \quad \frac{dY}{dx}(0) = 0, \quad Y(l) = Y_0, \quad \frac{dY}{dx}(l) = 0$$

This leads to $\quad Y(x) = \frac{3\,Y_0}{l^2}\,x^2 - \frac{2\,Y_0}{l^3}\,x^3$

$$y(x,t) = Y_0 \left(3\,\frac{x^2}{l^2} - 2\,\frac{x^3}{l^3} \right) \cos(\omega_n t - \phi) \qquad (E_4)$$

$$\text{Maximum strain energy} = \frac{1}{2}\,EI \int_0^l \left(\frac{\partial^2 y}{\partial x^2} \right)^2 dx \Bigg|_{max}$$

$$= \frac{6\,EI\,Y_0^2}{l^3} \qquad (E_5)$$

$$\text{Max. kinetic energy} = \frac{1}{2}\,M\,\omega_n^2\,Y_0^2 + \frac{1}{2}\left(\frac{m}{l}\right) Y_0^2\,\omega_n^2 \int_0^l \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3} \right)^2 dx$$

$$= \frac{1}{2}\,\omega_n^2\,Y_0^2 \left(M + \frac{13}{35}\,m \right) \qquad (E_6)$$

$$\therefore \quad m_{eff\,2} = \frac{13}{35}\,m = 0.3714\,m$$

(2.39) Stiffnesses of segments:

$$A_1 = \frac{\pi}{4}\,(D_1^2 - d_1^2) = \frac{\pi}{4}\,(2^2 - 1.75^2) = 0.7363 \text{ in}^2$$

$$k_1 = \frac{A_1\,E_1}{L_1} = \frac{(0.7363)\,(10^7)}{12} = 61.3583\,(10^4) \text{ lb/in}$$

$$A_2 = \frac{\pi}{4}\,(D_2^2 - d_2^2) = \frac{\pi}{4}\,(1.5^2 - 1.25^2) = 0.5400 \text{ in}^2$$

$$k_2 = \frac{A_2\,E_2}{L_2} = \frac{(0.5400)\,(10^7)}{10} = 54.0\,(10^4) \text{ lb/in}$$

$$A_3 = \frac{\pi}{4}\,(D_3^3 - d_3^2) = \frac{\pi}{4}\,(1^2 - 0.75^2) = 0.3436 \text{ in}^2$$

$$k_3 = \frac{A_3\,E_3}{L_3} = \frac{(0.3436)\,(10^7)}{8} = 42.9516\,(10^4) \text{ lb/in}$$

Equivalent stiffness (springs in series):

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

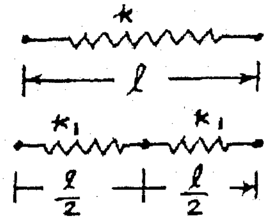
$$= 0.0162977\,(10^{-4}) + 0.0185185\,(10^{-4}) + 0.0232820\,(10^{-4}) = 0.0580982\,(10^{-4})$$

$$\text{or} \quad k_{eq} = 17.2122\,(10^4) \text{ lb/in}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_{eq}\,g}{W}} = \sqrt{\frac{17.2122\,(10^4)\,(386.4)}{10}} = 2578.9157 \text{ rad/sec}$$

(2.40)



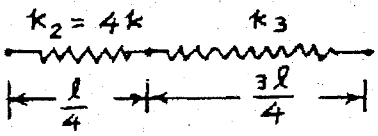
$$\frac{1}{k_{total}} = \frac{1}{k_1} + \frac{1}{k_1}$$

$$k_{total} = \frac{k_1}{2} \equiv k \; ; \; k_1 = 2k$$

$$\tau_n = 2\pi \sqrt{\frac{m}{k_{eq}}}$$

$$0.5 = 2\pi \sqrt{\frac{m}{4k}}$$

$$\sqrt{\frac{m}{k}} = \frac{1}{2\pi}$$



$$\frac{1}{k_{total}} = \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{4k} + \frac{1}{k_3} = \frac{1}{k}$$

$$k_3 = \frac{4}{3} k$$

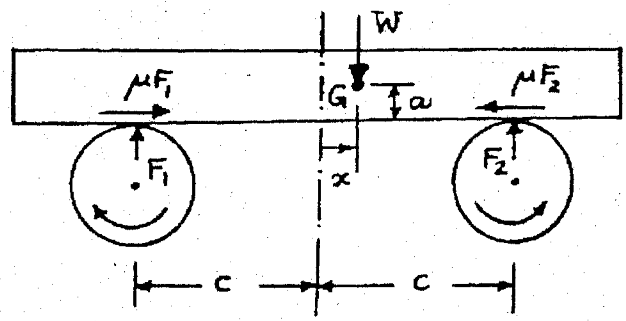
$$\tau_n = 2\pi \sqrt{\frac{m}{k_{eq}}} \quad \text{where} \quad k_{eq} = 4k + \frac{4}{3}k = \frac{16}{3}k$$

$$\therefore \tau_n = 2\pi \sqrt{\frac{3m}{16k}} = \frac{2\pi\sqrt{3}}{4}\sqrt{\frac{m}{k}} = \frac{2\pi\sqrt{3}}{4}\left(\frac{1}{2\pi}\right) = 0.4330 \text{ sec}$$

(2.41) Let μ = coefficient of friction

x = displacement of C.G. of block

F_1, F_2 = net reactions between roller and block on left and right sides.



Reactions at left and right due to static load W are $W(c-x)/2c$ and $W(c+x)/2c$, respectively.

M = moment about G due to motion of block = $(\mu F_2 - \mu F_1)a$

Reactions at left and right to balance $M = \frac{M}{2c} = \frac{\mu a}{2c}(F_2 - F_1)$

$$F_1 = \frac{W(c-x)}{2c} - \frac{\mu a}{2c}(F_2 - F_1) \; ; \quad F_2 = \frac{W(c+x)}{2c} + \frac{\mu a}{2c}(F_2 - F_1)$$

Subtraction gives
$$F_2 - F_1 = \frac{Wx}{c} + \frac{\mu a}{c}(F_2 - F_1)$$

i.e.,
$$F_2 - F_1 = \frac{Wx}{c}\left(\frac{c}{c-\mu a}\right) = \frac{Wx}{c-\mu a}$$

Restoring force $= \mu(F_2 - F_1) = \left(\frac{\mu W x}{c - \mu a}\right)$

Equation of motion :
$$\frac{W}{g}\ddot{x} + \frac{\mu W}{(c-\mu a)} x = 0$$

$$\omega_n = \omega = \sqrt{\frac{\mu W g}{W(c-\mu a)}} = \sqrt{\frac{\mu g}{c - \mu a}}$$

Solving this, we get $\omega = \left[c\omega^2/(g + a\omega^2)\right]$

(2.42) From problem 2.41,

Restoring force without springs $= \mu (F_2 - F_1) = \dfrac{\mu W x}{c - \mu a}$

spring restoring force $= 2kx$

Total restoring force $= \dfrac{\mu W x}{c-\mu a} + 2kx$

Equation of motion: $\dfrac{W}{g}\ddot{x} + \left(\dfrac{\mu W}{c-\mu a} + 2k\right)x = 0$

$$\omega_n = \omega = \left\{ \frac{[\mu W + 2k(c-\mu a)]\, g}{(c-\mu a)\, W} \right\}^{1/2}$$

Solution of this equation gives

$$\mu = \left(\frac{\omega^2 W c - 2 k g c}{W g + W \omega^2 a - 2 k g a} \right)$$

(2.43) (a) Natural frequency of vibration of electromagnet (without the automobile):

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{10000.0\ (386.4)}{3000.0}} = 35.8887 \text{ rad/sec}$$

(b) When the automobile is dropped, the electromagnet moves up by a distance (x_0) from its static equilibrium position.

x_0 = static deflection (elongation of cable) under the weight of automobile

$$= \frac{W_{auto}}{k} = \frac{2000}{10000} = 0.2 \text{ in}$$

$\dot{x}_0 = 0$

Resultant motion of electromagnet (+x upwards):

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} = x_0 = 0.2$$

and $\phi_0 = \tan^{-1}\left(\dfrac{x_0\, \omega_n}{\dot{x}_0} \right) = \tan^{-1}(\infty) = 90°$

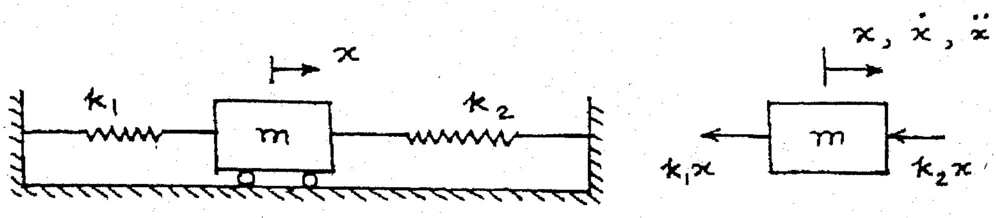
Hence $x(t) = 0.2 \sin(35.8887\, t + 90°) = 0.2 \cos 35.8887\, t$

(c) Maximum x(t):

$$x(t)\big|_{max} = A_0 = 0.2 \text{ in}$$

Maximum tension in cable during motion $= k\, x(t)\big|_{max}$ + Weigh of electromagnet $= 10000\ (0.2) + 3000 = 5{,}000$ lb.

2.44



(a) Newton's second law of motion:

$$F(t) = -k_1\, x - k_2\, x = m\, \ddot{x} \quad \text{or} \quad m\, \ddot{x} + (k_1 + k_2)\, x = 0$$

(b) D'Alembert's principle:

$$F(t) - m\, \ddot{x} = 0 \quad \text{or} \quad -k_1\, x - k_2\, x - m\, \ddot{x} = 0$$
$$\text{Thus} \quad m\, \ddot{x} + (k_1 + k_2)\, x = 0$$

(c) Principle of virtual work:

When mass m is given a virtual displacement δx,
Virtual work done by the spring forces = - $(k_1 + k_2)\, x\, \delta x$
Virtual work done by the inertia force = - $(m\, \ddot{x})\, \delta x$
According to the principle of virtual work, the total virtual work done by all forces must be equal to zero:

$$-m\, \ddot{x}\, \delta x - (k_1 + k_2)\, x\, \delta x = 0 \quad \text{or} \quad m\, \ddot{x} + (k_1 + k_2)\, x = 0$$

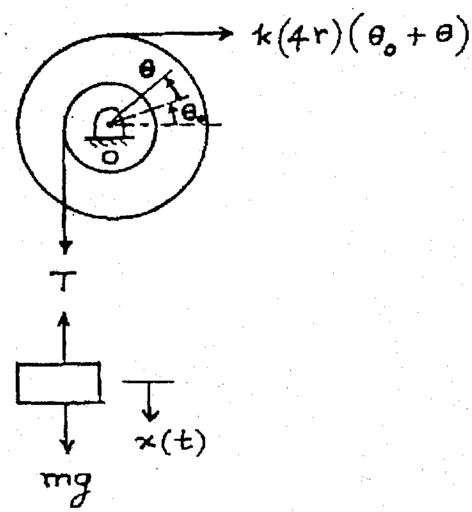
(d) Principle of conservation of energy:

$$T = \text{kinetic energy} = \frac{1}{2}\, m\, \dot{x}^2$$

$$U = \text{strain energy} = \text{potential energy} = \frac{1}{2}\, k_1\, x^2 + \frac{1}{2}\, k_2\, x^2$$

$$T + U = \frac{1}{2}\, m\, \dot{x}^2 + \frac{1}{2}\, (k_1 + k_2)\, x^2 = c = \text{constant}$$
$$\frac{d}{dt}\,(T + U) = 0 \quad \text{or} \quad m\, \ddot{x} + (k_1 + k_2)\, x = 0$$

2.45



Equation of motion:

$$\text{Mass m:} \quad m\,g - T = m\,\ddot{x} \tag{1}$$

$$\text{Pulley } J_0: \quad J_0\,\ddot{\theta} = T\,r - k\,4\,r\,(\theta + \theta_0)\,4\,r \tag{2}$$

where θ_0 = angular deflection of the pulley under the weight, mg, given by:

$$m\,g\,r = k\,(4\,r\,\theta_0)\,4\,r \quad \text{or} \quad \theta_0 = \frac{m\,g}{16\,r\,k} \tag{3}$$

Substituting Eqs. (1) and (3) into (2), we obtain

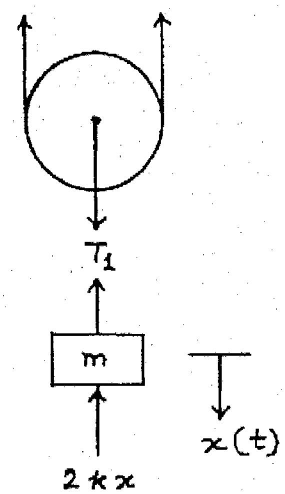
$$J_0\,\ddot{\theta} = (m\,g - m\,\ddot{x})\,r - k\,16\,r^2\,\left(\theta + \frac{m\,g}{16\,r\,k}\right) \tag{4}$$

Using $x = r\,\theta$ and $\ddot{x} = r\,\ddot{\theta}$, Eq. (4) becomes

$$(J_0 + m\,r^2)\,\ddot{\theta} + (16\,r^2\,k)\,\theta = 0$$

2.46 Consider the springs connected to the pulleys (by rope) to be in series. Then

$$\frac{1}{k_{eq}} = \frac{1}{k} + \frac{1}{5k} \quad \text{or} \quad k_{eq} = \frac{5}{6}\,k$$



Let the displacement of mass m be x. Then the extension of the rope (springs connected to the pulleys) = 2 x. From the free body diagram, the equation of motion of mass m:

$$m\,\ddot{x} + 2\,k\,x + k_{eq}\,(2\,x) = 0$$

$$\text{or} \quad m\,\ddot{x} + \frac{11}{3}\,k\,x = 0$$

2.47 T = kinetic energy = $T_{mass} + T_{pulley}$

$$= \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,J_0\,\dot{\theta}^2 = \frac{1}{2}\,(m\,r^2 + J_0)\,\dot{\theta}^2$$

$$U = \text{potential energy} = \frac{1}{2}\,k\,x_s^2 = \frac{1}{2}\,k\,(4\,r\,\theta)^2 = \frac{1}{2}\,k\,(16\,r^2)\,\theta^2$$

Using $\frac{d}{dt}\,(T + U) = 0$ gives

$$(m\,r^2 + J_0)\,\ddot{\theta} + (16\,r^2\,k)\,\theta = 0$$

2.48 $T = \text{kinetic energy} = \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,J_0\,\dot{\theta}^2$

$U = \text{potential energy} = \frac{1}{2}\,k\,x_s^2$

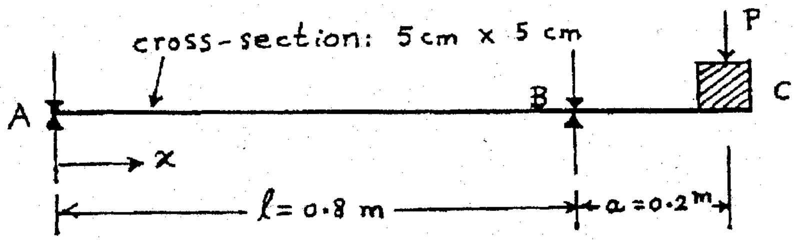
where $\theta = \frac{x}{r}$, x_s = extension of spring $= 4\, r\, \theta = 4\, x$. Hence

$$T = \frac{1}{2}\,(m + \frac{J_0}{r^2})\,\dot{x}^2 \;\; ; \;\; U = \frac{1}{2}\,(16\,k)\,x^2$$

Using tthe relation $\frac{d}{dt}\,(T + U) = 0$, we obtain the equation of motion of the system as:

$$(m + \frac{J_0}{r^2})\,\ddot{x} + 16\,k\,x = 0$$

2.49



Due to a load P at C, deflection at point C is given by (from Appendix B):

$$y(x) = \frac{P\,(x-\ell)}{6\,E\,I\,\ell}\left[a\,(3\,x-\ell)-(x-\ell)^2\right] \; ; \; \ell \le x \le \ell + a$$

$$y_C = y(x = \ell + a) = \frac{P\,a^2}{3\,E\,I\,\ell}\,(\ell + a)$$

Moment of inertia of cross section of beam:

$$I = \frac{1}{12}\,(0.05)\,(0.05)^3 = 52.0833\,(10^{-8})\ m^4$$

Equivalent stiffness:

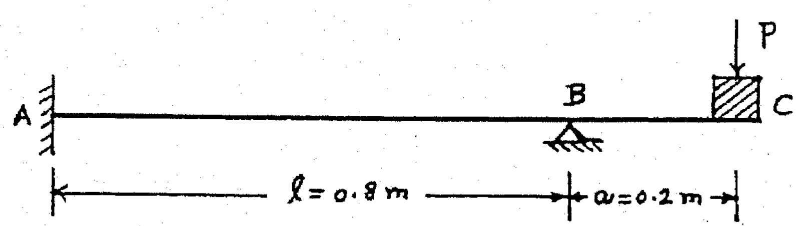
$$k_{eq} = \frac{P}{y_C} = \frac{3\,E\,I\,\ell}{a^2\,(\ell + a)} = \frac{3\,(207\,(10^9))\,(52.0833\,(10^{-8}))\,(0.8)}{(0.2)^2\,(0.8 + 0.2)}$$

$$= 6.4687\,(10^6)\ N/m$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{6.4687\,(10^6)}{50}} = 359.6872\ rad/sec$$

2.50



From Appendix B, the deflection of fixed-pinned beam with an overhang, due to load P at the free end, is given by:

$$y(x) = \frac{P\,a}{4\,E\,I\,\ell}\left[x^2 - \ell\,x^2 - \left(\frac{2\,\ell}{3\,a} + 1\right)(x-\ell)^3\right]; \quad \ell \le x \le \ell + a$$

Using $a = 0.2$, $\ell = 0.8$, $x = a + \ell = 1.0$, and

$$I = \frac{1}{12}\,(0.05)\,(0.05)^3 = 52.0833\,(10^{-8})\ \mathrm{m}^4$$

we obtain

$$y_C = \frac{P\,(0.2)}{4\,(207\,(10^9))\,(52.0833\,(10^{-8}))\,(0.8)}\left[1^2 - 0.8\,(1)^2 - \left(\frac{1.6}{0.6} + 1\right)(0.2)^3\right]$$

$$= P\,(9.895652\,(10^{-8}))$$

$$k_{eq} = \frac{P}{y_C} = 1010.5448\,(10^4)\ \mathrm{N/m}$$

$$\omega_n = \left\{\frac{k_{eq}}{m}\right\}^{\frac{1}{2}} = \left\{\frac{10.1054\,(10^6)}{50}\right\}^{\frac{1}{2}} = 449.5642\ \mathrm{rad/sec}$$

(2.51) The system of Fig. (A) can be drawn in equivalent form as shown in Fig. (B) where both pulleys have the same radius r_1. We notice in Fig. (B) that vibration can take place in only one way with one pulley moving clockwise and the other moving counter clockwise.

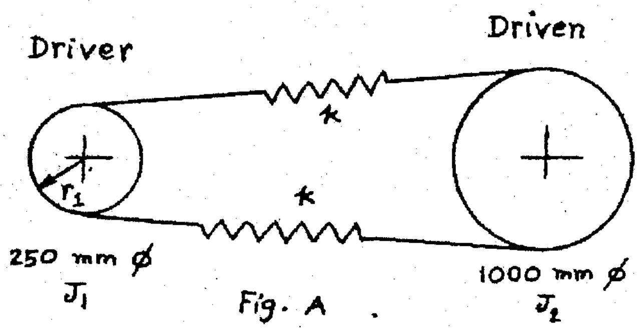


Fig. A.

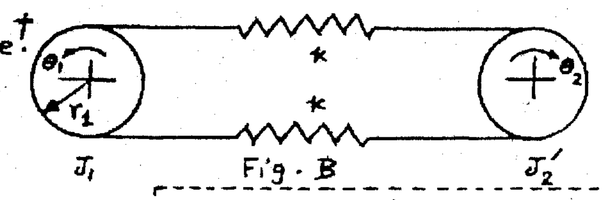


Fig. B

When pulleys rotate in opposite directions, $\dfrac{\theta_1}{\theta_2} = \dfrac{J_2}{J_1}$.

The spring force, which has the same value on either pulley is $-k_t(\theta_1+\theta_2)$ where k_t = torsional spring constant of the system. Equation of motion is

$$J_1\ddot{\theta}_1 + k_t(\theta_1+\theta_2) = 0 \quad \& \quad J_2\ddot{\theta}_2 + k_t(\theta_1+\theta_2) = 0$$

i.e. $J_1\ddot{\theta}_1 + k_t\left(1+\dfrac{J_1}{J_2}\right)\theta_1 = 0 \quad \& \quad J_2\ddot{\theta}_2 + k_t\left(\dfrac{J_2}{J_1}+1\right)\theta_2 = 0$

$$k_t = \frac{\Delta M_t}{\Delta\theta} = \left(\begin{matrix}\text{force in springs}\\ \text{due to } \Delta\theta\end{matrix}\right)\frac{r_1}{\Delta\theta}$$

$$= (2k\,r_1\,\Delta\theta)\frac{r_1}{\Delta\theta} = 2k\,r_1^2$$

$$= 2k\left(\frac{125}{1000}\right)^2 = k/32\ \ \mathrm{N\text{-}m/rad}$$

∴ Eq. (E_1) gives, for $\omega = 12\pi$ rad,

$k = 454.7935$ N/m.

Either of these equations gives

$$\omega = \left\{ k_t \left(\frac{J_1 + J_2}{J_1 J_2} \right) \right\}^{1/2} \quad --- (E_1)$$

Here $J_1 = 0.2/4 = 0.05$ kg-m^2,

$J_2' = J_2 (\text{speed ratio})^2 = 0.2 \left(\frac{1}{4}\right)^2 = 0.0125$ kg-m^2

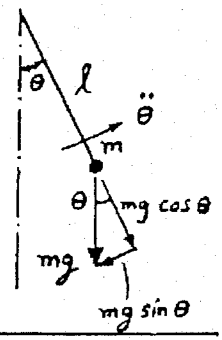
† The other possible motion is rotation of the two pulleys as a whole (as rigid body) in same direction. This will have a natural frequency of zero. See section 5.7.

(2.52) $ml\,\ddot{\theta} + mg \sin\theta = 0$

For small θ, $\quad ml\,\ddot{\theta} + mg\,\theta = 0$

$$\omega_n = \sqrt{\frac{g}{l}}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{9.81}{0.5}}} = 1.4185 \text{ sec}$$



(2.53) (a) $\omega_n = \sqrt{\frac{g}{l}}$

(b) $ml^2\ddot{\theta} + ka^2 \sin\theta + mgl \sin\theta = 0\,; \quad ml^2\ddot{\theta} + (ka^2 + mgl)\,\theta = 0$

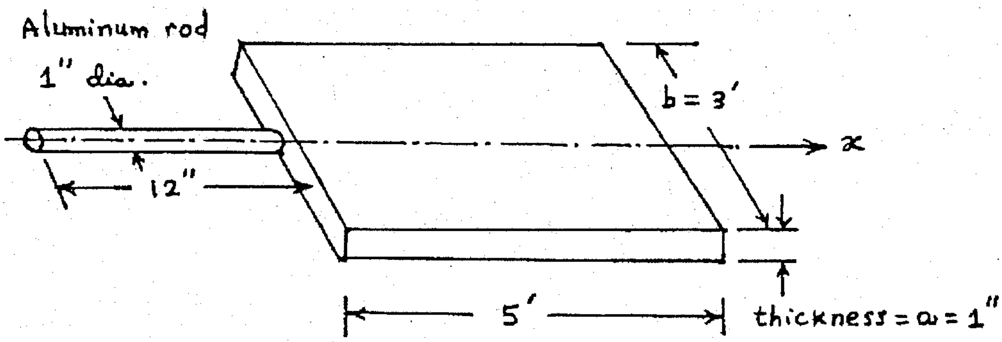
$$\omega_n = \sqrt{\frac{ka^2 + mgl}{ml^2}}$$

(c) $ml^2\ddot{\theta} + ka^2 \sin\theta - mgl \sin\theta = 0\,; \quad ml^2\ddot{\theta} + (ka^2 - mgl)\,\theta = 0$

$$\omega_n = \sqrt{\frac{ka^2 - mgl}{ml^2}}$$

configuration (b) has the highest natural frequency.

(2.54)



$$m = \text{mass of a panel} = (5 \times 12)\,(3 \times 12)\,(1)\,\left(\frac{0.283}{386.4}\right) = 1.5820$$

$$J_0 = \text{mass moment of inertia of panel about x-axis} = \frac{m}{12}\,(a^2 + b^2)$$

$$= \frac{1.5820}{12}\,(1^2 + 36^2) = 170.9878$$

$$I_0 = \text{polar moment of inertia of rod} = \frac{\pi}{32}\,d^4 = \frac{\pi}{32}\,(1)^4 = 0.098175 \text{ in}^4$$

$$k_t = \frac{G\, I_0}{\ell} = \frac{(3.8\,(10^6))\,(0.098175)}{12} = 3.1089\,(10^4)\text{ lb–in/rad}$$

$$\omega_n = \left\{\frac{k_t}{J_0}\right\}^{\frac{1}{2}} = \left\{\frac{3.1089\,(10^4)}{170.9878}\right\}^{\frac{1}{2}} = 13.4841\text{ rad/sec}$$

(2.55) I_0 = polar moment of inertia of cross section of shaft AB

$$= \frac{\pi}{32}\, d^4 = \frac{\pi}{32}\,(1)^4 = 0.098175\text{ in}^4$$

k_t = torsional stiffness of shaft AB $= \dfrac{G\, I_0}{\ell}$

$$= \frac{(12\,(10^6))\,(0.098175)}{6} = 19.635\,(10^4)\text{ lb–in/rad}$$

J_0 = mass moment of inertia of the three blades about y–axis

$$= 3\, J_0\,\big|_{PQ} = 3\left(\frac{1}{3}\, m\, \ell^2\right) = m\, \ell^2 = \left(\frac{2}{386.4}\right)(12)^2 = 0.7453$$

Torsional natural frequency:

$$\omega_n = \left\{\frac{k_t}{J_0}\right\}^{\frac{1}{2}} = \left\{\frac{19.635\,(10^4)}{0.7453}\right\}^{\frac{1}{2}} = 513.2747\text{ rad/sec}$$

(2.56) J_0 = mass moment of inertia of the ring = 1.0 kg–m².

I_{os} = polar moment of inertia of the cross section of steel shaft

$$= \frac{\pi}{32}\,(d_{os}^4 - d_{is}^4) = \frac{\pi}{4}\,(0.05^4 - 0.04^4) = 36.2266\,(10^{-8})\text{ m}^4$$

I_{ob} = polar moment of inertia of cross section of brass shaft

$$= \frac{\pi}{32}\,(d_{ob}^4 - d_{ib}^4) = \frac{\pi}{32}\,(0.04^4 - 0.03^4) = 17.1806\,(10^{-8})\text{ m}^4$$

k_{ts} = torsional stiffness of steel shaft

$$= \frac{G_s\, I_{os}}{\ell} = \frac{(80\,(10^9))\,(36.2266\,(10^{-8}))}{2} = 14490.64\text{ N–m/rad}$$

k_{tb} = torsional stiffness of brass shaft

$$= \frac{G_b\, I_{ob}}{\ell} = \frac{(40\,(10^9))\,(17.1806\,(10^{-8}))}{2} = 3436.12\text{ N–m/rad}$$

$k_{t_{eq}} = k_{ts} + k_{tb} = 17{,}926.76$ N–m/rad

Torsional natural frequency:

$$\omega_n = \sqrt{\frac{k_{t_{eq}}}{J_0}} = \sqrt{\frac{17926.76}{1}} = 133.8908\text{ rad/sec}$$

Natural time period:

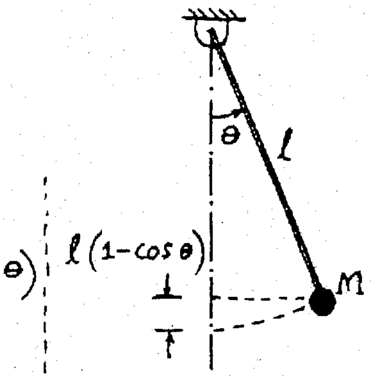
$$\tau_n = \frac{2\,\pi}{\omega_n} = \frac{2\,\pi}{133.8908} = 0.04693\text{ sec}$$

(2.57) Kinetic energy of system is

$$T = T_{rod} + T_{bob} = \frac{1}{2}\left(\frac{1}{3} m l^2\right)\dot{\theta}^2 + \frac{1}{2} M l^2 \dot{\theta}^2$$

Potential energy of system is (since mass of the rod acts through its center)

$$U = U_{rod} + U_{bob} = \frac{1}{2} m g l (1-\cos\theta) + \frac{1}{2} M g l (1-\cos\theta)$$



Equation of motion:

$$\frac{d}{dt}(T+U) = 0$$

i.e. $$\left(M + \frac{m}{3}\right) l^2 \ddot{\theta} + \left(M + \frac{m}{2}\right) g l \sin\theta = 0$$

For small angles, $$\ddot{\theta} + \frac{\left(M + \frac{m}{2}\right)}{\left(M + \frac{m}{3}\right)} \frac{g}{l} \cdot \theta = 0$$

$$\omega_n = \sqrt{\frac{\left(M + \frac{m}{2}\right) g}{\left(M + \frac{m}{3}\right) l}}$$

(2.58) For the shaft, $$J = \frac{\pi d^4}{32} = \frac{\pi (0.05)^4}{32} = 61.3594 \times 10^{-8} \text{ m}^4$$

$$k_t = \frac{GJ}{l} = \frac{(0.793 \times 10^{11})(61.3594 \times 10^{-8})}{2} = 24329.002 \text{ N-m/rad}$$

For the disc,

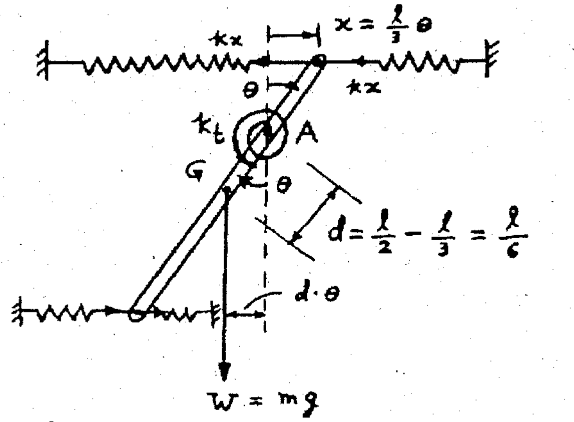
$$J_0 = \frac{M D^2}{8} = \left(\rho \frac{\pi D^2}{4} h\right)\frac{D^2}{8} = \frac{\rho \pi D^4 h}{32}$$

$$= \frac{(7.83 \times 10^3)\,\pi\,(1)^4 (0.1)}{32} = 76.8710 \text{ kg-m}^2$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left(\frac{24329.002}{76.8710}\right)^{1/2} = 17.7902 \text{ rad/sec}$$

(2.59) Equation of motion

$$J_A \ddot{\theta} = -W d\theta - 2k\left(\frac{l}{3}\theta\right)\frac{l}{3} - 2k\left(\frac{2l}{3}\theta\right)\frac{2l}{3} - k_t \theta$$



where

$$J_A = J_G + m d^2 = \frac{1}{12} m l^2 + m \frac{l^2}{36}$$

$$= \frac{1}{9} m l^2$$

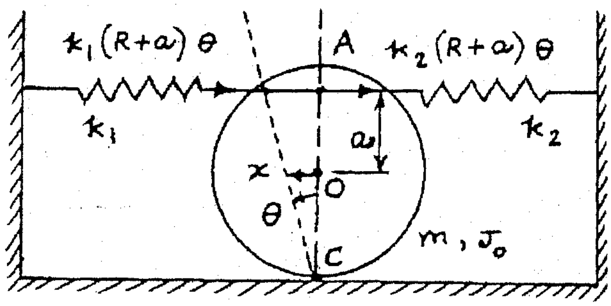
$$\therefore \frac{m l^2}{9}\ddot{\theta} + \left(m g d + 2k\frac{l^2}{9} + \frac{8 k l^2}{9} + k_t\right)\theta = 0$$

$$\omega_n = \sqrt{\frac{\left(m g d + \frac{2}{9} k l^2 + \frac{8}{9} k l^2 + k_t\right) 9}{m l^2}} = \sqrt{\frac{9 m g d + 10 k l^2 + 9 k_t}{m l^2}}$$

For given data,

$$\omega_n = \sqrt{\frac{9(10)(9.81)(5/6) + 10\,(2000)\,(5)^2 + 9\,(1000)}{10\,(5)^2}} = 45.1547 \ \frac{\text{rad}}{\text{sec}}$$

(2.60) $J_o = \frac{1}{2} m R^2$, $J_C = \frac{1}{2} m R^2 + m R^2$



Let angular displacement = θ

Equation of motion:

$$J_c\,\ddot{\theta} + k_1 (R+a)^2\,\theta + k_2 (R+a)^2\,\theta = 0$$

$$\omega_n = \sqrt{\frac{(k_1 + k_2)\,(R+a)^2}{J_C}} = \sqrt{\frac{(k_1 + k_2)(R+a)^2}{1.5\ m\ R^2}} \qquad (E_1)$$

Equation (E_1) shows that ω_n increases with the value of a.

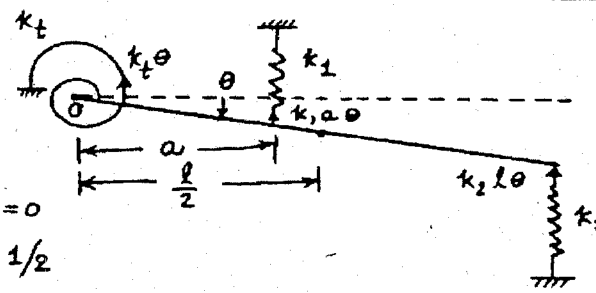
$\therefore$ ω_n will be maximum when $a = R$.

(2.61) Net g acting on the pendulum $= 9.81 - 5 = 4.81\ m/sec^2 = g_n$

$$\omega_n = \sqrt{\frac{g_n}{l}} = \sqrt{\frac{4.81}{5}} = 3.1016 \ \text{rad/sec}$$

$$\tau_n = 2\pi/\omega_n = 2.0258 \ \text{sec}$$

(2.62) Equation of motion:



$$J_o\,\ddot{\theta} = -k_t\theta - (k_1 a\theta)a - (k_2 l\theta)\,l$$

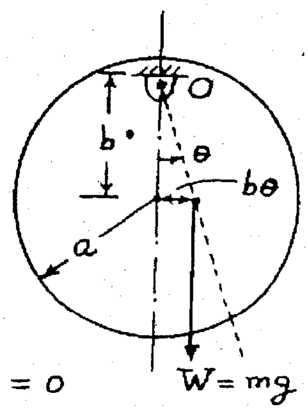
where $J_o = \frac{1}{12} m l^2 + m\left(\frac{l}{2}\right)^2 = \frac{1}{3} m l^2$

$$\therefore \quad \frac{1}{3} m l^2\,\ddot{\theta} + (k_t + k_1 a^2 + k_2 l^2)\,\theta = 0$$

$$\omega_n = \left\{\frac{3\,(k_t + k_1 a^2 + k_2 l^2)}{m l^2}\right\}^{1/2}$$

(2.63) $J_o = J_G + m b^2 = \frac{1}{2} m a^2 + m b^2$

Equation of motion:



$$J_o\,\ddot{\theta} + m g b\,\theta = 0$$

$$\omega_n = \sqrt{\frac{m g b}{J_o}} = \sqrt{\frac{2 g b}{a^2 + 2 b^2}}$$

$$\frac{\partial \omega_n}{\partial b} = \frac{1}{2}\left(\frac{2 g b}{a^2 + 2 b^2}\right)^{-\frac{1}{2}} \left\{\frac{(a^2 + 2b^2)(2g) - 2 g b\,(4b)}{(a^2 + 2 b^2)^2}\right\} = 0$$

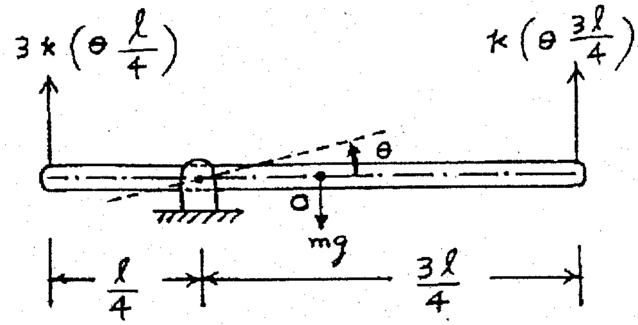
i.e., $b = \pm \dfrac{a}{\sqrt{2}}$

$$\omega_n \Big|_{b=+a/\sqrt{2}} = \sqrt{\frac{2g\,\frac{a}{\sqrt{2}}}{a^2 + 2\left(a^2/2\right)}} = \sqrt{\frac{g}{\sqrt{2}\,a}}$$

$b = -a/\sqrt{2}$ gives imaginary value for ω_n.

Since $\omega_n = 0$ when $b = 0$, we have $\omega_n\big|_{max}$ at $b = \dfrac{a}{\sqrt{2}}$.

(2.64)



Let θ be measured from static equilibrium position so that gravity force need not be considered.

(a) Newton's second law of motion:

$$J_0\,\ddot{\theta} = -3\,k\,(\theta\,\frac{\ell}{4})\,\frac{\ell}{4} - k\,(\theta\,\frac{3\,\ell}{4})\,(\frac{3\,\ell}{4}) \quad \text{or} \quad J_0\,\ddot{\theta} + \frac{3}{4}\,k\,\ell^2\,\theta = 0$$

(b) D'Alembert's principle:

$$M(t) - J_0\,\ddot{\theta} = 0 \quad \text{or} \quad -3\,k\,(\theta\,\frac{\ell}{4})\,(\frac{\ell}{4}) - k\,(\theta\,\frac{3\,\ell}{4})\,(\frac{3\,\ell}{4}) - J_0\,\ddot{\theta} = 0$$

$$\text{or} \quad J_0\,\ddot{\theta} + \frac{3}{4}\,k\,\ell^2\,\theta = 0$$

(c) Principle of virtual work:

Virtual work done by spring force:

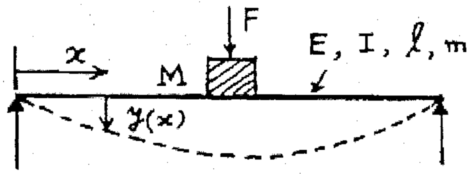
$$\delta W_s = -3\,k\,(\theta\,\frac{\ell}{4})\,(\frac{\ell}{4}\,\delta\theta) - k\,(\theta\,\frac{3\,\ell}{4})\,(\frac{3\,\ell}{4}\,\delta\theta)$$

Virtual work done by inertia moment $= -\,(J_0\,\ddot{\theta})\,\delta\theta$

Setting total virtual work done by all forces/moments equal to zero, we obtain

$$J_0\,\ddot{\theta} + \frac{3}{4}\,k\,\ell^2\,\theta = 0$$

(2.65) Let m_{eff} = effective part of mass of beam (m) at middle. Thus vibratory inertia force at middle is due to $(M + m_{eff})$. Assume a deflection shape: $y(x,t) = Y(x)\cos(\omega_n t - \phi)$ where $Y(x)$ = static deflection shape due to load at middle given by:



$$Y(x) = Y_0 \left(3 \frac{x}{\ell} - 4 \frac{x^3}{\ell^3}\right) \quad ; \quad 0 \le x \le \frac{\ell}{2}$$

where Y_0 = maximum deflection of the beam at middle $= \dfrac{F \ell^3}{48 E I}$

Maximum strain energy of beam = maximum work done by force F $= \dfrac{1}{2} F Y_0$.

Maximum kinetic energy due to distributed mass of beam:

$$= 2 \left\{ \frac{1}{2} \frac{m}{\ell} \int_0^{\frac{\ell}{2}} \dot{y}^2(x,t)\,|_{max}\, dx \right\} + \frac{1}{2} \left(\dot{y}_{max}\right)^2 M$$

$$= \frac{m\,\omega_n^2}{\ell} \int_0^{\frac{\ell}{2}} Y^2(x)\, dx + \frac{1}{2}\, \omega_n^2\, Y_{max}^2\, M$$

$$= \frac{m\,\omega_n^2}{\ell} \int_0^{\frac{\ell}{2}} Y_0^2 \left(\frac{9 x^2}{\ell^2} + 16 \frac{x^6}{\ell^6} - 24 \frac{x^4}{\ell^4}\right) dx + \frac{1}{2} Y_0^2\, M\, \omega_n^2$$

$$= \frac{m\,\omega_n^2\, Y_0^2}{\ell} \left[\frac{9}{\ell^2} \frac{x^3}{3} + \frac{16}{\ell^6} \frac{x^7}{7} - \frac{24}{\ell^4} \frac{x^5}{5}\right] \Big|_0^{\frac{\ell}{2}} + \frac{1}{2} Y_0^2\, M\, \omega_n^2$$

$$= \frac{1}{2} Y_0^2\, \omega_n^2 \left(\frac{17}{35} m + M\right)$$

This shows that $m_{eff} = \dfrac{17}{35} m = 0.4857\ m$

(2.66) For small angular rotation of bar PQ about P,

$$\frac{1}{2} (k_{12})_{eq} (\theta\, l_3)^2 = \frac{1}{2} k_1 (\theta\, l_1)^2 + \frac{1}{2} k_2 (\theta\, l_2)^2$$

$$(k_{12})_{eq} = \frac{k_1 l_1^2 + k_2 l_2^2}{l_3^2}$$

Since $(k_{12})_{eq}$ and k_3 are in series,

$$k_{eq} = \frac{(k_{12})_{eq}\ k_3}{(k_{12})_{eq} + k_3} = \frac{k_1 k_3 l_1^2 + k_2 k_3 l_2^2}{k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2}$$

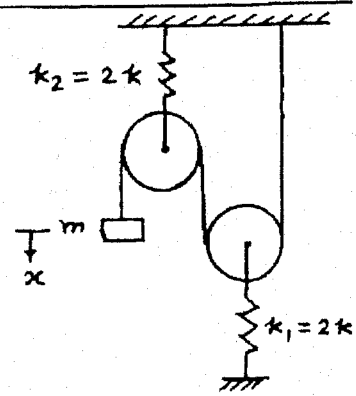
T = kinetic energy $= \frac{1}{2} m \dot{x}^2$, U = potential energy $= \frac{1}{2} k_{eq} x^2$

If $x = X \cos \omega_n t$,

$$T_{max} = \frac{1}{2} m\, \omega_n^2 X^2 \quad , \quad U_{max} = \frac{1}{2} k_{eq} X^2$$

$T_{max} = U_{max}$ gives $\qquad \omega_n = \sqrt{\dfrac{k_1 k_3 l_1^2 + k_2 k_3 l_2^2}{m(k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2)}}$

(2.67) When mass m moves by x,
spring k_1 deflects by $x/4$.



T = kinetic energy $= \frac{1}{2} m (\dot{x})^2$

U = potential energy $= 2\left\{\frac{1}{2}(2k)\left(\frac{x}{4}\right)^2\right\}$

$\quad = \frac{1}{8} k x^2$

For harmonic motion,

$$T_{max} = \frac{1}{2} m \omega_n^2 X^2, \qquad U_{max} = \frac{1}{8} k X^2$$

$T_{max} = U_{max}$ gives $\qquad \omega_n = \sqrt{\dfrac{k}{4m}}$

(2.68) Refer to the figure of solution of problem 2.24.

$T = \frac{1}{2} m \dot{x}^2 \quad , \qquad U = \frac{1}{2}\left[2k_1 (x \cos 45°)^2 + 2k_2 (x \cos 135°)^2\right]$

$\qquad\qquad\qquad\qquad = \frac{1}{2}(k_1 + k_2) x^2$

For harmonic motion,

$$T_{max} = \frac{1}{2} m \omega_n^2 X^2, \qquad U_{max} = \frac{1}{2}(k_1 + k_2) X^2$$

$T_{max} = U_{max}$ gives $\qquad \omega_n = \sqrt{\dfrac{k_1 + k_2}{m}}$

(2.69) Kinetic energy (K.E.) $= \frac{1}{2} m \dot{x}^2$

Potential energy (P.E.) $= \frac{1}{2} T_1 x + \frac{1}{2} T_2 x$ = work done in displacing mass m by distance x against the total force (tension) of $T_1 + T_2$.

$T_1 = \frac{x}{a} T, \qquad T_2 = \frac{x}{b} T \qquad$ from solution of problem 2.18

Max. K.E. $= \frac{1}{2} m \omega_n^2 X^2, \qquad$ Max. P.E. $= \frac{1}{2} T\left(\frac{1}{a} + \frac{1}{b}\right) X^2$

Max. K.E. = Max. P.E. gives $\qquad \omega_n = \sqrt{\dfrac{T(a+b)}{m\,a\,b}} = \sqrt{\dfrac{T l}{m\,a\,(l-a)}}$

(2.70) $T = K.E. = \frac{1}{2} J_A \dot{\theta}^2 = \frac{1}{2}(J_G + m d^2)\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{12} m l^2 + m \frac{l^2}{36}\right)\dot{\theta}^2$

$\qquad = \frac{1}{2}\left(\frac{m l^2}{9}\right)\dot{\theta}^2$

$U = P.E. = m g d (1 - \cos\theta) + 2\left(\frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2\right) + \frac{1}{2} k_t \theta^2$

$\qquad$ with $\cos\theta \simeq 1 - \frac{1}{2}\theta^2, \quad x_1 = \frac{l}{3}\theta$ and $x_2 = \frac{2l}{3}\theta$

$$U = mg\,\frac{l}{6}\,\frac{\theta^2}{2} + k\,\frac{l^2}{9}\,\theta^2 + k\,\frac{4l^2}{9}\,\theta^2 + \frac{1}{2}\,k_t\,\theta^2$$

$$T_{max} = \frac{1}{2}\left(\frac{ml^2}{9}\right)\Theta^2\,, \qquad U_{max} = \frac{1}{2}\,\frac{mgl}{6}\,\Theta^2 + \frac{1}{2}\left(\frac{10\,k\,l^2}{9}\right)\Theta^2 + \frac{1}{2}k_t\,\Theta^2$$

$T_{max} = U_{max}$ gives

$$\omega_n = \sqrt{\frac{mg\,\frac{l}{6} + \frac{10\,k\,l^2}{9} + k_t}{\frac{ml^2}{9}}} = 45.1547\ \frac{\text{rad}}{\text{sec}} \quad \text{for given data}$$

(2.71) Refer to the figure in the solution of problem 2.62.

$$T = \frac{1}{2}\,J_0\,\dot{\theta}^2$$

$$U = \frac{1}{2}\,k_t\,\theta^2 + \frac{1}{2}\,k_1\,(\theta a)^2 + \frac{1}{2}k_2\,(\theta\,l)^2$$

For $\theta(t) = \Theta\cos\omega_n t$,

$$T_{max} = \frac{1}{2}\,J_0\,\omega_n^2\,\Theta^2\,, \qquad U_{max} = \frac{1}{2}\left(k_t + k_1 a^2 + k_2 l^2\right)\Theta^2$$

$T_{max} = U_{max}$ gives

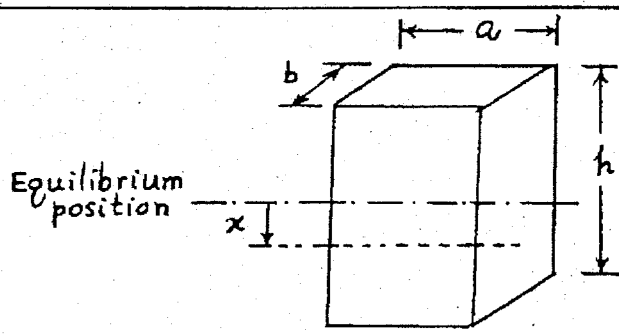
$$\omega_n = \sqrt{\frac{k_t + k_1 a^2 + k_2 l^2}{J_0}} = \sqrt{\frac{3\,(k_t + k_1 a^2 + k_2 l^2)}{m\,l^2}}$$

since $J_0 = ml^2/3$.

(2.72) When prism is displaced by x from equilibrium position, the weight of oil displaced

$= \rho_o\,g\,ab\,x$ = restoring force

Mass of prism = $m = \rho_w\,abh$



Equation of motion:

$$m\ddot{x} + \text{restoring force} = 0$$

$$\rho_w\,a\,b\,h\,\ddot{x} + \rho_o\,g\,a\,b\,x = 0$$

$$\omega_n = \sqrt{\frac{\rho_o\,g\,a\,b}{\rho_w\,a\,b\,h}} = \sqrt{\frac{\rho_o\,g}{\rho_w\,h}} \qquad (E_1)$$

Since ω_n is independent of cross-section of the prism, ω_n remains same even for a circular wooden prism.

(2.73)

$$T = \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,J_0\,\dot{\theta}^2 = \frac{1}{2}\left(m\,R^2 + \frac{1}{2}\,m\,R^2\right)\dot{\theta}^2$$

since $x = R\,\theta$ and $J_0 = \frac{1}{2}\,m\,R^2$.

$$U = \frac{1}{2}\,k_1\,x_1^2 + \frac{1}{2}\,k_2\,x_1^2 = \frac{1}{2}\,(k_1 + k_2)\,(R + a)^2\,\theta^2$$

where $x_1 = (R + a)\,\theta$. Using $\frac{d}{dt}\,(T + U) = 0$, we obtain

$$\left(\frac{3}{2}\, m\, R^2\right) \ddot{\theta} + (k_1 + k_2)\,(R + a)^2\,\theta = 0$$

(2.74) Let x(t) be measured from static equilibrium position of mass. T = kinetic energy of the system:

$$T = \frac{1}{2}\, m\, \dot{x}^2 + \frac{1}{2}\, J_0\, \dot{\theta}^2 = \frac{1}{2}\left(m + \frac{J_0}{r^2}\right) \dot{x}^2$$

since $\dot{\theta} = \frac{\dot{x}}{r}$ = angular velocity of pulley. U = potential energy of the system:

$$U = \frac{1}{2}\, k\, y^2 = \frac{1}{2}\, k\,(16\, x^2)$$

since $y = \theta\,(4\, r) = 4\, x$ = deflection of spring. $\frac{d}{dt}\,(T + U) = 0$ leads to:

$$m\,\ddot{x} + \frac{J_0}{r^2}\,\ddot{x} + 16\, k\, x = 0$$

This gives the natural frequency:

$$\omega_n = \sqrt{\frac{16\, k\, r^2}{m\, r^2 + J_0}}$$

(2.75) For pendulum, $\omega_n = \sqrt{g/l}$ in vaccum $= 0.5$ Hz $= \pi$ rad/sec

$$l = g/\pi^2 = 9.81/\pi^2 = 0.9940 \text{ m}$$

$\omega_d = \omega_n \sqrt{1 - \zeta^2}$ in viscous medium $= 0.45$ Hz $= 0.9\,\pi$ rad/sec

$$\zeta^2 = \frac{\omega_n^2 - \omega_d^2}{\omega_n^2} = \pi^2\left(\frac{1 - 0.81}{1}\right) = 1.8752$$

$$\zeta = 1.3694$$

Equation of motion: $m l^2\, \ddot{\theta} + c_t\, \dot{\theta} + m g l\, \theta = 0$

$$c_{ct} = 2\,(m l^2)\,\omega_n = 2\,(1)\,(0.994)^2 (\pi) = 6.2080$$

Since $\zeta = \frac{c_t}{c_{ct}} = 1.3694$, $c_t = 8.5013$ N-m-sec/rad.

(2.76) From Eq. (2.85),

$$\ln\left(\frac{x_j}{x_{j+1}}\right) = \ln(18) \Rightarrow \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 2.8904$$

$$\zeta = \left\{\frac{(2.8904)^2}{(2.8904)^2 + 4\,\pi^2}\right\}^{\frac{1}{2}} = 0.4179$$

(a) If damping is doubled, $\zeta_{new} = 0.8358$

$$\ln\left(\frac{x_j}{x_{j+1}}\right) = \frac{2\pi\,\zeta_{new}}{\sqrt{1-\zeta_{new}^2}} = \frac{2\pi(0.8358)}{\sqrt{1-(0.8358)^2}} = 9.5656$$

$$\therefore \quad \frac{x_j}{x_{j+1}} = 14265.362$$

(b) If damping is halved, $\zeta = 0.2090$

$$\ln\left(\frac{x_j}{x_{j+1}}\right) = \frac{2\pi\,\zeta_{new}}{\sqrt{1-\zeta_{new}^2}} = \frac{2\pi(0.2090)}{\sqrt{1-(0.2090)^2}} = 1.3428$$

$$\therefore \quad \frac{x_j}{x_{j+1}} = 3.8296$$

2.77 $x(t) = X\, e^{-\zeta\omega_n t} \sin\omega_d t$ where $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

For maximum or minimum of $x(t)$,

$$\frac{dx}{dt} = X\, e^{-\zeta\omega_n t}\left(-\zeta\omega_n \sin\omega_d t + \omega_d \cos\omega_d t\right) = 0$$

As $e^{-\zeta\omega_n t} \neq 0$ for finite t,

$$-\zeta\omega_n \sin\omega_d t + \omega_d \cos\omega_d t = 0$$

i.e. $\quad \tan\omega_d t = \sqrt{1-\zeta^2}/\zeta$

Using the relation

$$\sin\omega_d t = \pm\frac{\tan\omega_d t}{\sqrt{1+\tan^2\omega_d t}} = \frac{\pm\left(\sqrt{1-\zeta^2}/\zeta\right)}{\sqrt{1+\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)^2}} = \pm\sqrt{1-\zeta^2}$$

we obtain

$\sin\omega_d t = \sqrt{1-\zeta^2}$, $\quad \cos\omega_d t = \zeta$

and

$\sin\omega_d t = -\sqrt{1-\zeta^2}$, $\quad \cos\omega_d t = -\zeta$

$$\frac{d^2x}{dt^2} = X\, e^{-\zeta\omega_n t}\left[\zeta^2\omega_n^2 \sin\omega_d t - 2\zeta\omega_n\omega_d \cos\omega_d t - \omega_d^2 \sin\omega_d t\right]$$

When $\sin\omega_d t = \sqrt{1-\zeta^2}$ and $\cos\omega_d t = \zeta$,

$$\frac{d^2x}{dt^2} = -X\, e^{-\zeta\omega_n t}\,\omega_n^2\sqrt{1-\zeta^2} < 0$$

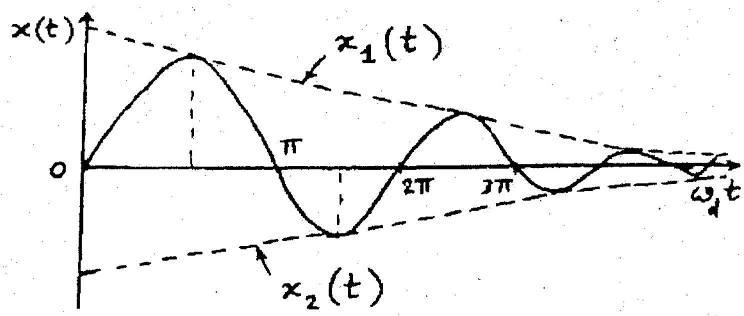
$\therefore \quad \sin\omega_d t = \sqrt{1-\zeta^2}$ corresponds to maximum of $x(t)$.

When $\sin\omega_d t = -\sqrt{1-\zeta^2}$ and $\cos\omega_d t = -\zeta$,

$$\frac{d^2x}{dt^2} = X\, e^{-\zeta\omega_n t}\, \omega_n^2 \sqrt{1-\zeta^2} \quad > 0$$

$\therefore \sin\omega_d t = -\sqrt{1-\zeta^2}$ corresponds to minimum of $x(t)$.

Enveloping curves:



Let the curve passing through the maximum (or minimum) points be

$$x(t) = C\, e^{-\zeta\omega_n t}$$

For maximum points, $t_{max} = \dfrac{\sin^{-1}\left(\sqrt{1-\zeta^2}\right)}{\omega_d}$

and

$$C\, e^{-\zeta\omega_n t_{max}} = X\, e^{-\zeta\omega_n t_{max}} \sin\omega_d t_{max}$$

i.e.

$$C = X\sqrt{1-\zeta^2}$$

$$\therefore x_1(t) = X\sqrt{1-\zeta^2}\; e^{-\zeta\omega_n t}$$

Similarly for minimum points, $t_{min} = \dfrac{\sin^{-1}\left(-\sqrt{1-\zeta^2}\right)}{\omega_d}$

and

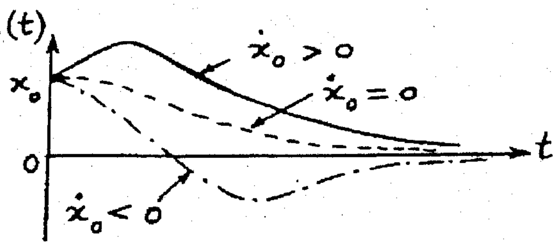
$$C\, e^{-\zeta\omega_n t_{min}} = X\, e^{-\zeta\omega_n t_{min}} \sin\omega_d t_{min}$$

i.e.

$$C = -X\sqrt{1-\zeta^2}$$

$$\therefore x_2(t) = -X\sqrt{1-\zeta^2}\; e^{-\zeta\omega_n t}$$

(2.78) $x(t) = \left[x_0 + (\dot{x}_0 + \omega_n x_0)t\right] e^{-\omega_n t}$ ---- (E_1)



For $x_0 > 0$, graph of Eq. (E_1) is shown for different $\dot{x}_0$. We assume $\dot{x}_0 > 0$ as it is the only case that gives a maximum.

For maximum of $x(t)$,

$$\frac{dx}{dt} = e^{-\omega_n t}\left\{-(\dot{x}_0 + \omega_n x_0)\,\omega_n t + \dot{x}_0\right\} = 0$$

$$t_m = \frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)} \quad ---- (E_2)$$

$$\frac{d^2x}{dt^2} = -e^{-\omega_n t}\left\{2\,\omega_n\,\dot{x}_0 + \omega_n^2\,x_0 - \omega_n^2\,(\dot{x}_0 + \omega_n\,x_0)\,t\right\} \quad ---- (E_3)$$

(E_2) and (E_3) give

$$\left.\frac{d^2x}{dt^2}\right|_{t=t_m} = -e^{-\omega_n t_m}\left\{2\omega_n\,\dot{x}_0 + \omega_n^2\,x_0 - \omega_n^2\,(\dot{x}_0 + \omega_n\,x_0)\,t_m\right\}$$

$$= -e^{-\omega_n\left(\frac{\dot{x}_0}{\omega_n(\dot{x}_0+\omega_n x_0)}\right)}\left\{\omega_n\dot{x}_0 + \omega_n^2\,x_0\right\} \quad ---- (E_4)$$

For $x_0 > 0$ and $\dot{x}_0 > 0$, $\left.\frac{d^2x}{dt^2}\right|_{t_m} < 0$

Hence t_m given by Eq. (E_2) corresponds to a maximum of $x(t)$.

$$x\Big|_{t=t_m} = \left\{x_0 + (\dot{x}_0 + \omega_n\,x_0)\,\frac{\dot{x}_0}{\omega_n\,(\dot{x}_0 + \omega_n\,x_0)}\right\} e^{-\omega_n t_m}$$

$$= \left(x_0 + \frac{\dot{x}_0}{\omega_n}\right) e^{-\left(\frac{\dot{x}_0}{\dot{x}_0 + \omega_n x_0}\right)} \quad ---- (E_5)$$

(2.79) Equation (2.92) can be expressed as $\delta = \frac{1}{m}\,\ln\left(\frac{x_0}{x_m}\right)$

For half cycle, $m = \frac{1}{2}$ and hence $\delta = 2\,\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right) = 2\,\ln\left(\frac{1}{0.15}\right)$

$= 3.7942$

Necessary damping ratio ζ_0 is

$$\zeta_0 = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{3.7942^2}{\sqrt{4\pi^2 + 3.7942^2}}$$

$= 0.5169$

(a)

If $\zeta = \frac{3}{4}\,\zeta_0 = 0.3877$, the overshoot can be determined by finding δ from Eq. (2.85):

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi\,(0.3877)}{\sqrt{1 - 0.3877^2}} = 2.6427 = 2\,\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right)$$

$$\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right) = 1.32135$$

$$x_{\frac{1}{2}} = x_0 \big/ e^{1.32135} = 0.266775\;x_0$$

$\therefore$ Overshoot is $26.6775\,\%$

(b)

If $\zeta = \frac{5}{4}\,\zeta_0 = 0.6461$, δ is given by

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi\,(0.6461)}{\sqrt{1-(0.6461)^2}} = 5.3189 = 2\,\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right)$$

$$\frac{x_0}{x_{\frac{1}{2}}} = 14.2888 \quad , \qquad x_{\frac{1}{2}} = 0.0700\ x_0$$

$\therefore$ overshoot $= 7\%$

(2.80) (i) (a) Viscous damping, (b) Coulomb damping.

(iii) (a) $\tau_d = 0.2$ sec, $f_d = 5$ Hz, $\omega_d = 31.416$ rad/sec.
(b) $\tau_n = 0.2$ sec, $f_n = 5$ Hz, $\omega_n = 31.416$ rad/sec.

(ii) (a) $\dfrac{x_i}{x_{i+1}} = e^{\zeta\,\omega_n\,\tau_d}$

$$\ln\left(\frac{x_i}{x_{i+1}}\right) = \ln 2 = 0.6931 = \frac{2\,\pi\,\zeta}{\sqrt{1-\zeta^2}}.\text{sp}$$

or $\quad 39.9590\ \zeta^2 = 0.4804 \quad$ or $\quad \zeta = 0.1096$

Since $\omega_d = \omega_n \sqrt{1-\zeta^2}$, we find

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{31.416}{\sqrt{0.98798}} = 31.6065 \text{ rad/sec}$$

$$k = m\,\omega_n^2 = \left(\frac{500}{9.81}\right)(31.6065)^2 = 5.0916\ (10^4) \text{ N/m}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\,m\,\omega_n}$$

Hence $c = 2\,m\,\omega_n\,\zeta = 2\,(\dfrac{500}{9.81})\,(31.6065)\,(0.1096) = 353.1164$ N–s/m

(b) From Eq. (2.116):

$$k = m\,\omega_n^2 = \frac{500}{9.81}\,(31.416)^2 = 5.0304\ (10^4) \text{ N/m}$$

Using N = W = 500 N,

$$\mu = \frac{0.002\ k}{4\ W} = \frac{(0.002)\,(5.0304\,(10^4))}{4\,(500)} = 0.0503$$

(2.81) (a) $c_c = 2\sqrt{k\,m} = 2\sqrt{5000 \times 50} = 1000$ N-s/m

(b) $c = c_c/2 = 500$ N-s/m

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m}}\ \sqrt{1-\left(\frac{c}{c_c}\right)^2} = \sqrt{\frac{5000}{50}}\ \sqrt{1-\left(\frac{1}{2}\right)^2}$$

$= 8.6603$ rad/sec

(c) From Eq. (2.85), $\quad \delta = \dfrac{2\pi}{\omega_d}\left(\dfrac{c}{2m}\right) = \dfrac{2\pi}{8.6603}\left(\dfrac{500}{2 \times 50}\right)$

$= 3.6276$

(2.82) $m = 2000$ kg, $\quad v = \dot{x}_0 = 10$ m/sec, $\quad k = 40{,}000$ N/m
$c = 20{,}000$ N-sec/m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{40000}{2000}} = 4.4721 \text{ rad/sec}$$

$$c_c = 2\sqrt{km} = 25,298.221 \text{ N-sec/m}$$

$$\zeta = c/c_c = 0.7906$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 4.4721 \sqrt{1-(0.7906)^2} = 2.7384 \text{ rad/sec}$$

$$\tau_d = 2\pi/\omega_d = 2.2945 \text{ sec}$$

(a) For $x_o = 0$ and $\dot{x}_o = 10$ m/sec, Eq. (2.72) gives

$$x(t) = e^{-\zeta\omega_n t} \frac{\dot{x}_o}{\omega_n\sqrt{1-\zeta^2}} \sin \omega_n \sqrt{1-\zeta^2}\, t$$

At x_{max}, $\omega_n t \simeq \frac{\pi}{2}$ and $\sin \omega_n\sqrt{1-\zeta^2}\, t \simeq 1$

$$\therefore x_{max} \simeq e^{-0.7906(\pi/2)} \cdot \left(\frac{10}{2.7384}\right)\cdot(1) = 1.0548 \text{ m}$$

(b) $t = \tau_d/4 = 2.2945/4 = 0.5736$ sec.

(2.83) $\omega_n = 200$ cycles/min $= 20.944$ rad/sec, $\omega_d = 180$ cycles/min $= 18.8496 \frac{\text{rad}}{\text{sec}}$

$J_o = 0.2$ kg-m^2

Since $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$, $\zeta = \sqrt{1-\left(\frac{\omega_d}{\omega_n}\right)^2} = \sqrt{1-\left(\frac{18.8496}{20.944}\right)^2} = 0.4359$

$$= \frac{c_t}{(c_t)_{cri}} = \frac{c_t}{2 J_o \omega_n}$$

$$c_t = 2 J_o \omega_n \zeta = 2(0.2)(20.944)(0.4359)$$

$$= 3.6518 \text{ N-m-s/rad}$$

Eq. (2.72) can be used to obtain $\theta(t)$ for $\dot{\theta}_o = 0$, $\theta_o = 2^\circ = 0.03491$ rad and $t = \tau_d = \frac{2\pi}{\omega_d} = 0.3333$ sec,

$$\theta(t) = e^{-\zeta\omega_n t}\,\theta_o \left\{\cos \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d t\right\}$$

$$= e^{-(0.4359)(20.944)(0.3333)}(0.03491)\left\{\cos 18.8496 \times 0.3333 + \frac{0.4359 \times 20.944}{18.8496} \sin 18.8496 \times 0.3333\right\}$$

$$= 0.001665 \text{ rad} = 0.09541^\circ$$

(2.84) Assume that the bicycle and the boy fall as a rigid body by 5 cm at point A. Thus the mass (m_{eq}) will be subjected to an initial downward displacement of 5 cm (t = 0 assumed at point A):

$$x_0 = 0.05 \text{ m}, \ \dot{x}_0 = 0$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{(50000)(9.81)}{800}} = 24.7614 \text{ rad/sec}$$

$$c_c = 2\, m\, \omega_n = 2 \left(\frac{800}{9.81} \right) (24.7614) = 4038.5566 \text{ N-s/m}$$

$$\zeta = \frac{c}{c_c} = \frac{1000.0}{4038.5566} = 0.2476 \text{ (underdamping)}$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 24.7614 \sqrt{1-0.2476^2} = 23.9905 \text{ rad/sec}$$

Response of the system:

$$x(t) = X\, e^{-\zeta \omega_n t} \sin(\omega_d t + \phi)$$

$$\text{where} \quad X = \left\{ x_0^2 + \left(\frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d} \right)^2 \right\}^{\frac{1}{2}}$$

$$= \left\{ (0.05)^2 + \left(\frac{(0.2476)\,(24.7614)\,(0.05)}{23.9905} \right)^2 \right\}^{\frac{1}{2}} = 0.051607 \text{ m}$$

$$\text{and} \quad \phi = \tan^{-1} \left(\frac{x_0\, \omega_d}{\dot{x}_0 + \zeta \omega_n x_0} \right) = \tan^{-1} \left(\frac{0.05\,(23.9905)}{0.2476\,(24.7614)\,(0.05)} \right) = 75.6645^\circ$$

Thus the displacement of the boy (positive downward) in vertical direction is given by

$$x(t) = 0.051607\, e^{-6.1309\, t} \sin(23.9905\, t + 75.6645^\circ) \text{ m}$$

2.85 Reduction in amplitude of viscously damped free vibration in one cycle = 0.5 in.

$$\frac{x_1}{x_2} = \frac{6.0}{5.5} = 1.0909; \quad \ln \frac{x_1}{x_2} = 0.08701 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\text{i.e.,} \quad 0.007571\,(1-\zeta^2) = 39.478602\, \zeta^2 \quad \text{or} \quad \zeta = 0.013847$$

2.86 $\tau_d = 0.2 \text{ sec} = \frac{2\pi}{\omega_d}$, $\omega_d = 31.416$ rad/sec

From Eq. (2.92) $\delta = \frac{1}{50} \ln 10 = 0.04605$

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.04605}{\sqrt{(2\pi)^2 + 0.04605^2}} = 0.007329$$

When damping is neglected.

$$\omega_n = \omega_d / \sqrt{1-\zeta^2} = 31.417 \text{ rad/sec} \; ; \quad \tau_n = \frac{2\pi}{\omega_n} = 0.19999 \text{ sec}$$

Proportional decrease in period $= \left(\frac{0.2 - 0.19999}{0.2} \right) = 0.00005$

2.87 For critically damped system, Eq. (2.80) gives

$$x(t) = \{ x_0 + (\dot{x}_0 + \omega_n x_0)\, t \}\, e^{-\omega_n t} \qquad (E_1)$$

$$\dot{x}(t) = e^{-\omega_n t} \{ \dot{x}_0 - \dot{x}_0\, \omega_n t - \omega_n^2\, x_0 t \} \qquad (E_2)$$

Let t_m = time at which $x = x_{max}$ and $\dot{x} = 0$ occur.

Here $x_0 = 0$ and $\dot{x}_0$ = initial recoil velocity. By setting $\dot{x}(t) = 0$, Eq. (E_2) gives

$$t_m = \frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)} = \frac{\dot{x}_0}{\omega_n \dot{x}_0} = \frac{1}{\omega_n} \qquad (E_3)$$

With Eq. (E_3) for t_m and $x_0 = 0$, (E_1) gives

$$x_{max} = \dot{x}_0 t_m e^{-\omega_n t_m} = \frac{\dot{x}_0 e^{-1}}{\omega_n}$$

i.e. $$\dot{x}_0 = \omega_n x_{max} e = \omega_n (0.5)(2.7183) \qquad (E_4)$$

Using $\dot{x}_0 = 10$ m/sec, $\omega_n = 10/(0.5 * 2.7183) = 7.3575 \frac{rad}{sec}$

When mass of gun is 500 kg,

the stiffness of the spring is

$$k = \omega_n^2 m = (7.3575)^2 (500) = 27{,}066.403 \text{ N/m}$$

2.88 $k = 5000$ N/m , $c_c = 0.2$ N-s/mm $= 200$ N-s/m

$= 2\sqrt{k m} = 2\sqrt{5000\, m}$

$m = 2$ kg

$\omega_n = \sqrt{k/m} = \sqrt{5000/2} = 50$ rad/sec

Logarithmic decrement $= \delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 2.0$

i.e., $\zeta = \frac{c}{c_c} = 0.3033$ and $c = 0.3033\,(0.2) = 60.66$ N-s/m

Assuming $x_0 = 0$ and $\dot{x}_0 = 1$ m/s,

$$x(t) = e^{-\zeta\omega_n t} \frac{\dot{x}_0}{\omega_n\sqrt{1-\zeta^2}} \sin\sqrt{1-\zeta^2}\,\omega_n t$$

For x_{max}, $\omega_n t \approx \pi/2$ and $\sin\sqrt{1-\zeta^2}\,\omega_n t \approx 1$

$$\therefore \quad x_{max} \simeq e^{-0.3033(\pi/2)} \frac{1}{50\sqrt{1-0.3033^2}} (1) = 0.01303 \text{ m}$$

2.89 For an overdamped system, Eq. (2.81) gives

$$x(t) = e^{-\zeta\omega_n t}\left(C_1 e^{\omega_d t} + C_2 e^{-\omega_d t}\right) \qquad (E_1)$$

Using the relations $$e^{\pm x} = \cosh x \pm \sinh x \qquad (E_2)$$

Eq. (E_1) can be rewritten as

$$x(t) = e^{-\zeta\omega_n t}\left(C_3 \cosh\omega_d t + C_4 \sinh\omega_d t\right) \qquad (E_3)$$

where $C_3 = C_1 + C_2$ and $C_4 = C_1 - C_2$.

Differentiating (E_3),

$$\dot{x}(t) = e^{-\zeta\omega_n t}\left[C_3\,\omega_d \sinh \omega_d t + C_4\,\omega_d \cosh \omega_d t\right]$$
$$- \zeta\omega_n\, e^{-\zeta\omega_n t}\left[C_3 \cosh \omega_d t + C_4 \sinh \omega_d t\right] \qquad (E_4)$$

Initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$ with (E_3) and (E_4) give

$$C_3 = x_0\,, \quad C_4 = (\dot{x}_0 + \zeta\omega_n x_0)/\omega_d \qquad (E_5)$$

Thus (E_3) becomes

$$x(t) = x_0\, e^{-\zeta\omega_n t}\left(\cosh \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sinh \omega_d t\right)$$
$$+ \frac{\dot{x}_0}{\omega_d}\, e^{-\zeta\omega_n t} \sinh \omega_d t \qquad (E_6)$$

(i) When $\dot{x}_0 = 0$, Eq. (E_6) gives

$$x(t) = x_0\, e^{-\zeta\omega_n t}\left(\cosh \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sinh \omega_d t\right) \qquad (E_7)$$

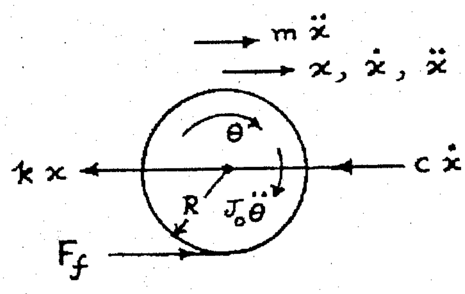
since $e^{-\zeta\omega_n t}$, $\cosh \omega_d t$, $\frac{\zeta\omega_n}{\omega_d}$ and $\sinh \omega_d t$ do not change sign (always positive) and $e^{-\zeta\omega_n t}$ approaches zero with increasing t, $x(t)$ will not change sign.

(ii) When $x_0 = 0$, Eq. (E_6) gives

$$x(t) = \frac{\dot{x}_0}{\omega_d}\, e^{-\zeta\omega_n t} \sinh \omega_d t \qquad (E_8)$$

Here also, ω_d, $e^{-\zeta\omega_n t}$ and $\sinh \omega_d t$ do not change sign (always positive) and $e^{-\zeta\omega_n t}$ approaches zero with increasing t, $x(t)$ will not change sign.

2.90 Newton's second law of motion:



$$\sum F = m\,\ddot{x} = -k\,x - c\,\dot{x} + F_f \qquad (1)$$
$$\sum M = J_0\,\ddot{\theta} = -F_f\,R \qquad (2)$$

where F_f = friction force.

Using $J_0 = \dfrac{m\,R^2}{2}$ and $\ddot{\theta} = \dfrac{\ddot{x}}{R}$, Eq. (2) gives

$$F_f = -\frac{1}{2\,R}\left(m\,R^2\right)\frac{\ddot{x}}{R} = -\frac{1}{2}\,m\,\ddot{x} \qquad (3)$$

Substitution of Eq. (3) into (1) yields:

$$\frac{3}{2}\,m\,\ddot{x} + c\,\dot{x} + k\,x = 0 \tag{4}$$

The undamped natural frequency is: $\omega_n = \sqrt{\dfrac{2\,k}{3\,m}}$ (5)

(2.91) Newton's second law of motion: (measuring x from static equilibrium position of cylinder)

$$\sum F = m\,\ddot{x} = -\,k\,x - c\,\dot{x} - k\,x + F_f \tag{1}$$

$$\sum M = J_0\,\ddot{\theta} = -\,F_f\,R \tag{2}$$

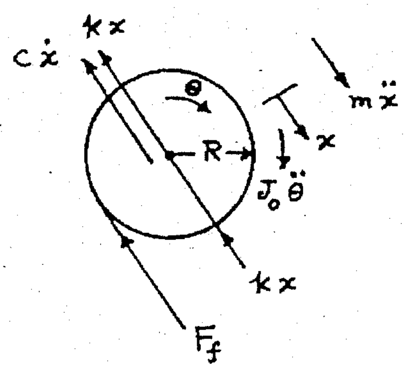
where F_f = friction force. Using $J_0 = \frac{1}{2}\,m\,R^2$ and $\ddot{\theta} = \frac{\ddot{x}}{R}$, Eq. (2) gives

$$F_f = -\,\frac{1}{2}\,m\,\ddot{x} \tag{3}$$

Substitution of Eq. (3) into (1) gives

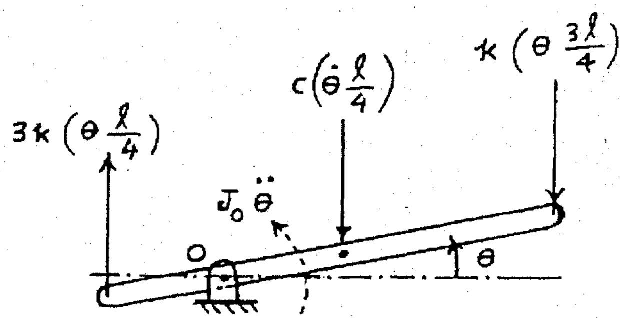
$$\frac{3}{2}\,m\,\ddot{x} + c\,\dot{x} + 2\,k\,x = 0 \tag{4}$$

Undamped natural frequency of the system:



$$\omega_n = \sqrt{\frac{4\,k}{3\,m}} \tag{4}$$

(2.92)



Consider a small angular displacement of the bar θ about its static equilibrium position. Newton's second law gives:

$$\sum M = J_0\,\ddot{\theta} = -k\left(\theta\,\frac{3\,\ell}{4}\right)\left(\frac{3\,\ell}{4}\right) - c\left(\dot{\theta}\,\frac{\ell}{4}\right)\left(\frac{\ell}{4}\right) - 3\,k\left(\theta\,\frac{\ell}{4}\right)\left(\frac{\ell}{4}\right)$$

$$\text{i.e.,}\quad J_0\,\ddot{\theta} + \frac{c\,\ell^2}{16}\,\dot{\theta} + \frac{3}{4}\,k\,\ell^2\,\theta = 0$$

where $J_0 = \frac{7}{48} m \ell^2$. The undamped natural frequency of torsional vibration is given by:

$$\omega_n = \sqrt{\frac{3 k \ell^2}{4 J_0}} = \sqrt{\frac{36 k}{7 m}}$$

(2.93) Let δx = virtual displacement given to cylinder. Virtual work done by various forces:

Inertia forces: $\delta W_i = -(J_0 \ddot{\theta}) (\delta\theta) - (m \ddot{x}) \delta x = -(J_0 \ddot{\theta}) (\frac{\delta x}{R}) - (m \ddot{x}) \delta x$

Spring force: $\delta W_s = -(k x) \delta x$
Damping force: $\delta W_d = -(c \dot{x}) \delta x$
By setting the sum of virtual works equal to zero, we obtain:

$$-\frac{J_0}{R} \left(\frac{\ddot{x}}{R} \right) - m \ddot{x} - k x - c \dot{x} = 0 \quad \text{or} \quad \frac{3}{2} m \ddot{x} + c \dot{x} + k x = 0$$

(2.94) Let δx = virtual displlacement given to cyllinder from its static equillibrium position. Virtualll works done by various forces:

Inertia forces: $\delta W_i = -(J_0 \ddot{\theta}) \delta\theta - (m \ddot{x}) \delta\theta = -(J_0 \frac{\ddot{x}}{R}) (\frac{\delta x}{R}) - (m \ddot{x}) \delta x$

Spring force: $\delta W_s = -(k x) \delta x - (k x) \delta x = -2 k x \delta x$

Damping force: $\delta W_d = -(c \dot{x}) \delta x$
By setting the sum of virtual works equal to zero, we find

$$-\frac{J_0}{R} \frac{\ddot{x}}{R} - m \ddot{x} - 2 k x - c \dot{x} = 0 \qquad (1)$$

Using $J_0 = \frac{1}{2} m R^2$, Eq. (1) can be rewritten as

$$\frac{3}{2} m \ddot{x} + c \dot{x} + 2 k x = 0 \qquad (2)$$

(2.95) See figure given in the solution of Problem 2.92. Let $\delta\theta$ be virtuall angular displacement given to the bar about its static equilibrium position. Virtual works done by various forces:
Inertia force: $\delta W_i = -(J_0 \ddot{\theta}) \delta\theta$
Spring forces:

$$\delta W_s = -(k \theta \frac{3 \ell}{4}) (\frac{3 \ell}{4} \delta\theta) - (3 k \theta \frac{\ell}{4}) (\frac{\ell}{4} \delta\theta) = -\left(\frac{3}{4} k \ell^2 \theta \right) \delta\theta$$

Damping force: $\delta W_d = -(c \dot{\theta} \frac{\ell}{4}) (\frac{\ell}{4} \delta\theta)$

By setting the sum of virtual works equal to zero, we get the equation of motion as:

$$J_0 \ddot{\theta} + c \frac{\ell^2}{16} \dot{\theta} + \frac{3}{4} k \ell^2 \theta = 0$$

(2.96) See solution of Problem 2.72. When wooden prism is given a displacement x, equation of motion becomes: $m\ddot{x}$ + restoring force = 0
where m = mass of prism = 40 kg and restoring force = weight of fluid displaced $= \rho_0\, g\, a\, b\, x = \rho_0\,(9.81)\,(0.4)\,(0.6)\,x = 2.3544\,\rho_0\,x$ where ρ_0 is the density of the fluid. Thus the equation of motion becomes:

$$40\,\ddot{x} + 2.3544\,\rho_0\,x = 0$$

$$\text{Natural frequency} = \omega_n = \sqrt{\frac{2.3544\,\rho_0}{40}}$$

$$\text{Since } \tau_n = \frac{2\,\pi}{\omega_n} = 0.5, \text{we find}$$

$$\omega_n = \frac{2\,\pi}{0.5} = 4\,\pi = \sqrt{\frac{2.3544\,\rho_0}{40}}$$

Hence $\rho_0 = 2682.8816\ \text{kg/m}^3$.

(2.97) Let x = displacement of mass and P = tension in rope on the left of mass. Equations of motion:

$$\sum F = m\,\ddot{x} = -k\,x - P \quad \text{or} \quad P = -m\,\ddot{x} - k\,x \tag{1}$$

$$\sum M = J_0\,\ddot{\theta} = P\,r_2 - c\,(\dot{\theta}\,r_1) \tag{2}$$

Using Eq. (1) in (2), we obtain

$$-(m\,\ddot{x} + k\,x)\,r_2 - c\,\dot{\theta}\,r_1 = J_0\,\ddot{\theta} \tag{3}$$

With $x = \theta\,r_2$, Eq. (3) can be written as:

$$(J_0 + m\,r_2^2)\,\ddot{\theta} + c\,r_1\,\dot{\theta} + k\,r_2^2\,\theta = 0 \tag{4}$$

For given data, Eq. (4) becomes

$$[5 + 10\,(0.25)^2]\,\ddot{\theta} + c\,(0.1)\,\dot{\theta} + k\,(0.25)^2\,\theta = 0$$

$$\text{or} \quad 5.625\,\ddot{\theta} + 0.1\,c\,\dot{\theta} + 0.0625\,k\,\theta = 0 \tag{5}$$

Since amplitude is reduced by 80% in 10 cycles,

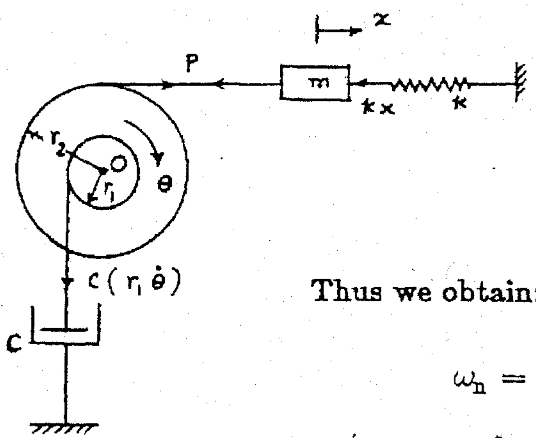
$$\frac{x_1}{x_{11}} = \frac{1.0}{0.2} = 5 = e^{10\,\zeta\,\omega_n\,\tau_d}$$

$$\ln\frac{x_1}{x_{11}} = \ln 5 = 1.6094 = 10\,\zeta\,\omega_n\,\tau_d \tag{6}$$

Since the natural frequency (assumed to be undamped torsional vibration frequency) is 5 Hz, $\omega_n = 2\,\pi\,(5) = 31.416$ rad/sec. Also

$$\tau_d = \frac{1}{f_d} = \frac{2\,\pi}{\omega_d} = \frac{2\,\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{0.2}{\sqrt{1-\zeta^2}} \tag{7}$$

Eq. (6) gives



$$1.6094 = 10\,\zeta\,(31.416)\left(\frac{0.2}{\sqrt{1-\zeta^2}}\right) = \frac{62.832\,\zeta}{\sqrt{1-\zeta^2}}$$

i.e., $\sqrt{1-\zeta^2} = \dfrac{62.832}{1.6094}\,\zeta = 39.0406\,\zeta$

i.e., $\zeta = 0.02561$

Thus we obtain:

$$\omega_n = \sqrt{\frac{0.0625\ k}{5.625}} = 31.416 \quad \text{or} \quad k = 8.8827\ (10^4)\ \text{N/m}$$

$$\zeta = 0.02561 = \frac{c}{c_c} = \frac{c}{2\ m_{eq}\ \omega_n} = \frac{0.10\ c}{2\ (5.625)\ (31.416)}$$

or $c = 90.5134$ N–s/m

(2.98) $m = 20$ kg, $k = 4000$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{20}} = 14.1421 \text{ rad/sec}$$

Amplitudes of successive cycles : 50, 45, 40, 35 mm

Amplitudes of successive cycles diminish by 5 mm $= 5 \times 10^{-3}$ m

System has Coulomb damping.

$$\frac{4\mu N}{k} = 5 * 10^{-3} \Rightarrow \mu N = \left\{\frac{(5\times10^{-3})(4000)}{4}\right\} = 5 \text{ N}$$

= damping force

Frequency of damped vibration = 14.1421 rad/sec.

(2.99) $m = 20$ kg, $k = 10000$ N/m, $\dfrac{4\mu N}{k} = \dfrac{150-100}{4}$ mm $= 12.5 \times 10^{-3}$ m

$$\mu = \frac{(12.5\times10^{-3})(10\,000)}{4\,(20\times 9.81)} = 0.1593$$

Time elapsed $= 4\,\tau_n = 4 \times \dfrac{2\pi}{\omega_n} = 8\pi\sqrt{\dfrac{m}{k}} = 1.124$ sec

---

(2.100) $m = 10$ kg, $k = 3000$ N/m, $\mu = 0.12$, $X = 100$ mm

$$\frac{4\mu N}{k} = \frac{4\,(0.12)\,(10\times9.81)}{3000} = 0.0157 \text{ m} = 15.7 \text{ mm}$$

As $6\left(\dfrac{4\mu N}{k}\right) = 94.2$ mm, mass comes to rest at $(100 - 94.2) = 5.8$ mm

---

(2.101) $mg = 25$ N, $k = 1000$ N/m, damping force = constant

Mass released with $x_0 = 10$ cm and $\dot{x}_0 = 0$.

Static deflection of spring due to self weight of mass $= \dfrac{25}{1000}$
$= 0.025$ m

at $t = 0$: $x = 0.1$ m, $\dot{x} = 0$

$x_0 = 0.1$

$$x_1 = x_0 - 2\,\frac{\mu N}{k}\ , \quad x_2 = x_0 - \frac{4\,\mu N}{k}$$

$$x_3 = a_0 - \frac{6\,\mu N}{k}\ , \quad x_4 = x_0 - \frac{8\,\mu N}{k} = 0$$

i.e., $x_0 = \dfrac{8\,\mu N}{k} = 0.1$

Magnitude of damping force $= \mu N = \dfrac{x_0 k}{8} = \dfrac{(0.1)(1000)}{8}$
$= 12.5$ N

(2.102) $m = 20$ kg, $k = 10{,}000$ N/m, $\mu N = 50$ N, $x_0 = 0.05$ m

(a) Number of half cycles elapsed before mass comes to rest (r) is given by:

$$r \geq \left\{ \frac{x_0 - \frac{\mu N}{k}}{2\,\frac{\mu N}{k}} \right\} = \frac{0.05 - \left(\frac{50}{10000}\right)}{2\left(\frac{50}{10\,000}\right)} = 4.5$$

$\therefore\ r = 5$

(b) Time elapsed before mass comes to rest:

$$t_p = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{20}{10000}} = 0.2810 \text{ sec}$$

Time taken $= (2.5 \text{ cycles})\, t_p = 0.7025$ sec

(c) Final extension of spring after 5 half-cycles:

$$x_5 = 0.05 - 5\left(\frac{2\,\mu N}{k}\right) = 0.05 - 5\left(2 * \frac{50}{10\,000}\right) = 0$$

(displacement from static equilibrium position $= 0$)

But static deflection $= \dfrac{mg}{k} = \dfrac{20 * 9.81}{10\,000} = 0.01962$ m

$\therefore$ Final extension of spring $= 1.9620$ cm.

(2.103) (a) Equation of motion for angular oscillations of pendulum:

$$J_0\,\ddot{\theta} + mgl\sin\theta \pm mg\mu\,\frac{d}{2}\cos\theta = 0$$

For small angles, $\ddot{\theta} + \dfrac{mgl}{J_0}\left(\theta \pm \dfrac{\mu d}{2l}\right) = 0$

This shows that the angle of swing decreases by $\left(\dfrac{\mu d}{2l}\right)$ in each quarter cycle.

(b) For motion from right to left:

$$\theta(t) = A_1\cos\omega_n t + A_2\sin\omega_n t + \frac{\mu d}{2l}$$

where $\omega_n = \sqrt{\dfrac{mgl}{J_0}}$.

Let $\theta(t=0) = \theta_0$ and $\dot{\theta}(t=0) = 0$. Then $A_1 = \theta_0 - \dfrac{\mu d}{2l}$, $A_2 = 0$

$$\theta(t) = \left(\theta_0 - \frac{\mu d}{2\ell}\right) \cos \omega_n t + \frac{\mu d}{2\ell}$$

For motion from left to right:

$$\theta(t) = A_3 \cos \omega_n t + A_4 \sin \omega_n t - \frac{\mu d}{2\ell}$$

At $\omega_n t = \pi$, $\theta = -\theta_0 + \frac{2\mu d}{2\ell}$, $\dot{\theta} = 0$ from previous solution.

$$A_3 = \theta_0 - \frac{3\mu d}{2\ell}, \quad A_4 = 0$$

$$\theta(t) = \left(\theta_0 - \frac{3\mu d}{2\ell}\right) \cos \omega_n t - \frac{\mu d}{2\ell}$$

(c) The motion ceases when $\left(\theta_0 - n\,\frac{4\mu d}{2\ell}\right) < \frac{\mu d}{2\ell}$

where n denotes the number of cycles.

2.104 $x(t) = X \sin \omega t$ (under sinusoidal force $F_0 \sin \omega t$)

Damping force $= \mu N$

Total displacement per cycle $= 4X$

Energy dissipated per cycle $= \Delta W = 4 \mu N X$ $\quad (E_1)$

If c_{eq} = equivalent viscous damping constant, energy dissipated per cycle is given by Eq. (2.98):

$$\Delta W = \pi\, c_{eq}\, \omega X^2 \qquad (E_2)$$

Equating (E_1) and (E_2) gives

$$c_{eq} = \frac{4\mu N X}{\pi \omega X^2} = \frac{4\mu N}{\pi \omega X} \qquad (E_3)$$

2.105 Due to viscous damping:

$$\delta = \ln\left(\frac{X_m}{X_{m+1}}\right) = 2\pi \zeta$$

z_1 = percent decrease in amplitude per cycle at X_m

$$= 100\left(\frac{X_m - X_{m+1}}{X_m}\right) = 100\left(1 - \frac{X_{m+1}}{X_m}\right) = 100\left(1 - e^{-2\pi\zeta}\right)$$

Due to Coulomb damping:

z_2 = percent decrease in amplitude per cycle at X_m

$$= 100\left(\frac{X_m - X_{m+1}}{X_m}\right) = 100\left(\frac{4\mu N}{k X_m}\right)$$

When both types of damping are present:

$$z_1 + z_2 \Big|_{X_m = 20\text{ mm}} = 2 \quad ; \quad z_1 + z_2 \Big|_{X_m = 10\text{ mm}} = 3$$

i.e.,

$$100\left(1-e^{-2\pi\zeta}\right)+\frac{400}{0.02}\left(\frac{\mu N}{k}\right)=2$$

$$100\left(1-e^{-2\pi\zeta}\right)+\frac{400}{0.01}\left(\frac{\mu N}{k}\right)=3$$

The solution of these equations gives

$$50\left(1-e^{-2\pi\zeta}\right)=0.5 \quad \text{and} \quad \frac{\mu N}{k}=0.5\times 10^{-6}\ \text{m}$$

(1.106) Coulomb damping.

(a) Natural frequency $= \omega_n = \frac{2\pi}{\tau_n} = \frac{2\pi}{1} = 6.2832$ rad/sec. Reduction in amplitude in each cycle:

$$= \frac{4\mu N}{k} = 4\mu g\frac{m}{k} = \frac{4\mu g}{\omega_n^2} = 4\mu\left(\frac{9.81}{6.2832^2}\right)$$

$$= 0.9940\,\mu = \frac{0.5}{100} = 0.005\ \text{m}$$

Kinetic coefficient of friction $= \mu = 0.00503$

(b) Number of half-cycles executed (r) is:

$$r \geq \frac{\left(x_0-\frac{\mu N}{k}\right)}{\left(\frac{2\mu N}{k}\right)} = \frac{\left(x_0-\frac{\mu g}{\omega_n^2}\right)}{\left(\frac{2\mu g}{\omega_n^2}\right)}$$

$$\geq \frac{\left(0.1-\frac{0.00503\,(9.81)}{6.2832^2}\right)}{\left(\frac{2\,(0.00503)\,(9.81)}{6.2832^2}\right)}$$

$$\geq 39.5032$$

$$\geq 40$$

Thus the block stops oscillating after 20 cycles.

(2.107) Friction force $= \mu N = 0.2\ (5) = 1$ N. $k = \frac{25}{0.10} = 250$ N/m. Reduction in amplitude in each cycle $= \frac{4\mu N}{k} = \frac{4\,(1)}{250} = 0.016$ m. Number of half-cycles executed before the motion ceases (r):

$$r \geq \left(\frac{x_0-\frac{\mu N}{k}}{\frac{2\mu N}{k}}\right) = \frac{0.1-0.004}{0.008} \geq 12$$

Thus after 6 cycles, the mass stops at a distance of 0.1 - 6 (0.016) = 0.004 m from the unstressed position of the spring.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{250\ (9.81)}{5}} = 22.1472 \text{ rad/sec}$$

$$\tau_n = \frac{2\ \pi}{\omega_n} = 0.2837 \text{ sec}$$

Thus total time of vibration $= 6\ \tau_n = 1.7022$ sec.

(2.108) Energy dissipated in each full load cycle is given by the area enclosed by the hysteresis loop.

The area can be found by counting the squares enclosed by the hysteresis loop. In Fig. 2.99, the number of squares is ≈ 33. Since each square $= \frac{100 \times 1}{1000} = 0.1$ N-m, the energy dissipated in a cycle is

$$\Delta W = 33 \times 0.1 = 3.3 \text{ N-m} = \pi\, k\, \beta\, X^2$$

Since the maximum deflection $= X = 4.3$ mm, and the slope of the force-deflection curve is

$$k \simeq \frac{1800 \text{ N}}{11 \text{ mm}} = 1.6364 \times 10^5 \text{ N/m},$$

the hysteresis damping constant β is given by

$$\beta = \frac{\Delta W}{\pi\, k\, X^2} = \frac{3.3}{\pi\,(1.6364 \times 10^5)\,(0.0043)^2} = 0.3472$$

$\delta = \pi \beta =$ logarithmic decrement $= \pi\,(0.3472) = 1.0908$

Equivalent viscous damping ratio $= \zeta_{eq} = \beta/2 = 0.1736$.

(2.109) $\frac{x_j}{x_{j+1}} = \frac{2 + \pi\beta}{2 - \pi\beta} = 1.1$, $\beta = 0.03032$

$$c_{eq} = \beta \sqrt{m k} = 0.03032 \sqrt{1 \times 2} = 0.04288 \text{ N-s/m}$$

$$\Delta W = \pi\, k\, \beta\, X^2 = \pi\,(2)\,(0.03032)\left(\frac{10}{1000}\right)^2 = 19.05 \times 10^{-6} \text{ N-m}$$

(2.110) Logarithmic decrement $= \delta = \ln\left(\frac{x_j}{x_{j+1}}\right) \simeq \pi\beta$

For n cycles, $\delta = \frac{1}{n} \ln\left(\frac{x_0}{x_n}\right) \simeq \pi\beta$

$$\frac{1}{100} \ln\left(\frac{30}{20}\right) = 0.004055 = \pi\beta$$

$\beta = 0.001291$

2.111 Torque $= 2 \times 10^{-3}$ N-m

angle $= 50° = 80$ divisions

For a torsional system, Eq. (2.84) gives

$$\frac{\theta_1}{\theta_2} = e^{\zeta \omega_n \tau_d} \qquad (E_1)$$

(b) For one cycle, $\tau_d = 2$ sec and (E_1) gives

$$\frac{80}{5} = e^{2 \zeta \omega_n} \quad \text{or} \quad \zeta \omega_n = \frac{1}{2} \ln (16) = 1.3863 \qquad (E_2)$$

Since $\tau_d = \dfrac{2\pi}{\sqrt{\omega_n^2 - \zeta^2 \omega_n^2}}$,

$$\omega_n^2 = \frac{(2\pi)^2}{\tau_d^2} + \zeta^2 \omega_n^2 = \frac{4\pi^2}{4} + 1.3863^2 = 11.7915$$

i.e., $\omega_n = 3.4339$ rad/sec $\qquad (E_3)$

(d) Since angular displacement of rotor under applied torque

$= 50° = 0.8727$ rad,

$k_t =$ torque/angular displacement $= 2 \times 10^{-3} / 0.8727$

$= 2.2917 \times 10^{-3}$ N-m/rad $\qquad (E_4)$

(a) Mass moment of inertia of rotor is

$$J_0 = \frac{k_t}{\omega_n^2} = 2.2917 \times 10^{-3} / 11.7915 = 1.9436 \times 10^{-4} \text{ N-m-s}^2 \qquad (E_5)$$

(c) $c_t = 2 J_0 \zeta \omega_n \qquad (E_6)$

Eqs. (E_2) and (E_3) give $\quad \zeta = \dfrac{\zeta \omega_n}{\omega_n} = \dfrac{1.3863}{3.4339} = 0.4037$

Eq. (E_6) gives $\quad c_t = 5.3887 \times 10^{-4}$ N-m-s/rad.

(2.112) The input data for Program 2 and results are given below.

```
C       THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
        DIMENSION X(50),XD(50),XDD(50),T(50)
        DATA M,K,C,X0,XD0,N,DELT/
      2 4.0,2500.0,0.0,0.1,-10.0,50,0.01/
C       END OF PROBLEM-DEPENDENT DATA
--------------------------------------------
FREE VIBRATION ANALYSIS
OF A SINGLE DEGREE OF FREEDOM SYSTEM

DATA

M     = 0.40000000E+01
K     = 0.25000000E+04
C     = 0.00000000E+00
X0    = 0.10000000E+00
XD0   =-0.10000000E+02
N     =    50
DELT  = 0.99999998E-02


SYSTEM IS UNDAMPED


RESULTS:
```

I	TIME(I)	X(I)	XD(I)	XDD(I)
1	0.100000E-01	-0.207036E-02	-0.103076E+02	0.129397E+01
2	0.200000E-01	-0.104012E+00	-0.997438E+01	0.650075E+02
3	0.300000E-01	-0.199487E+00	-0.902097E+01	0.124679E+03
4	0.400000E-01	-0.282558E+00	-0.750667E+01	0.176599E+03
5	0.500000E-01	-0.348062E+00	-0.552565E+01	0.217539E+03
6	0.600000E-01	-0.391924E+00	-0.320107E+01	0.244953E+03
7	0.700000E-01	-0.411419E+00	-0.677464E+00	0.257137E+03
8	0.800000E-01	-0.405334E+00	0.188826E+01	0.253334E+03
9	0.900000E-01	-0.374047E+00	0.433659E+01	0.233779E+03
10	0.100000E+00	-0.319503E+00	0.651528E+01	0.199690E+03
⋮				
46	0.460000E+00	0.398512E+00	-0.264442E+01	-0.249070E+03
47	0.470000E+00	0.359954E+00	-0.502704E+01	-0.224971E+03
48	0.480000E+00	0.299016E+00	-0.709711E+01	-0.186885E+03
49	0.490000E+00	0.219486E+00	-0.872591E+01	-0.137179E+03
50	0.500000E+00	0.126310E+00	-0.981218E+01	-0.789439E+02

(2.113) The problem-dependent data and results are given (given by Program 2).

```
C       THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
        DIMENSION X(50),XD(50),XDD(50),T(50)
        DATA M,K,C,X0,XD0,N,DELT/
      2 4.0,2500.0,100.0,0.1,-10.0,50,0.01/
C       END OF PROBLEM-DEPENDENT DATA
--------------------------------------------
FREE VIBRATION ANALYSIS
OF A SINGLE DEGREE OF FREEDOM SYSTEM
```

DATA

```
M     = 0.40000000E+01
K     = 0.25000000E+04
C     = 0.10000000E+03
X0    = 0.10000000E+00
XD0   =-0.10000000E+02
N     =   50
DELT  = 0.99999998E-02
```

SYSTEM IS UNDER DAMPED

RESULTS:

I	TIME(I)	X(I)	XD(I)	XDD(I)
1	0.100000E-01	-0.957279E-02	0.807168E+01	-0.195809E+03
2	0.200000E-01	0.613786E-01	0.612586E+01	-0.191508E+03
3	0.300000E-01	0.113259E+00	0.427345E+01	-0.177623E+03
4	0.400000E-01	0.147433E+00	0.259570E+01	-0.157038E+03
5	0.500000E-01	0.165937E+00	0.114627E+01	-0.132367E+03
6	0.600000E-01	0.171219E+00	-0.456091E-01	-0.105872E+03
7	0.700000E-01	0.165914E+00	-0.971334E+00	-0.794127E+02
8	0.800000E-01	0.152654E+00	-0.163885E+01	-0.544377E+02
9	0.900000E-01	0.133930E+00	-0.206856E+01	-0.319924E+02
10	0.100000E+00	0.111980E+00	-0.228942E+01	-0.127520E+02
⋮				
45	0.450000E+00	-0.113169E-03	-0.310021E-01	0.845783E+00
46	0.460000E+00	-0.381386E-03	-0.227080E-01	0.806065E+00
47	0.470000E+00	-0.569294E-03	-0.149992E-01	0.730789E+00
48	0.480000E+00	-0.684317E-03	-0.817052E-02	0.631961E+00
49	0.490000E+00	-0.736251E-03	-0.240284E-02	0.520228E+00
50	0.500000E+00	-0.736195E-03	0.222121E-02	0.404592E+00

(2.114) The problem-dependent data (to be used in Program 2) and output are given.

```
C     THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(50),XD(50),XDD(50),T(50)
      DATA M,K,C,X0,XD0,N,DELT/
     2 4.0,2500.0,200.0,0.1,-10.0,50,0.01/
C     END OF PROBLEM-DEPENDENT DATA
```

FREE VIBRATION ANALYSIS
OF A SINGLE DEGREE OF FREEDOM SYSTEM

DATA

```
M     = 0.40000000E+01
K     = 0.25000000E+04
C     = 0.20000000E+03
X0    = 0.10000000E+00
XD0   =-0.10000000E+02
N     =   50
DELT  = 0.99999998E-02
```

```
SYSTEM IS CRITICALLY DAMPED

RESULTS:

   I     TIME(I)          X(I)            XD(I)           XDD(I)

   1   0.100000E-01   0.194700E-01   -0.632776E+01   0.304219E+03
   2   0.200000E-01  -0.303265E-01   -0.379082E+01   0.208495E+03
   3   0.300000E-01  -0.590458E-01   -0.206660E+01   0.140234E+03
   4   0.400000E-01  -0.735759E-01   -0.919699E+00   0.919699E+02
   5   0.500000E-01  -0.787888E-01   -0.179066E+00   0.581963E+02
   6   0.600000E-01  -0.780956E-01    0.278913E+00   0.348641E+02
   7   0.700000E-01  -0.738539E-01    0.543044E+00   0.190065E+02
   8   0.800000E-01  -0.676676E-01    0.676676E+00   0.845846E+01
   9   0.900000E-01  -0.606045E-01    0.724619E+00   0.164687E+01
  10   0.100000E+00  -0.533553E-01    0.718244E+00  -0.256515E+01
   :
  46   0.460000E+00  -0.339359E-04    0.772422E-03  -0.174112E-01
  47   0.470000E+00  -0.270210E-04    0.616356E-03  -0.139296E-01
  48   0.480000E+00  -0.215048E-04    0.491539E-03  -0.111364E-01
  49   0.490000E+00  -0.171069E-04    0.391783E-03  -0.889736E-02
  50   0.500000E+00  -0.136023E-04    0.312109E-03  -0.710396E-02
```

(2.115) *The problem-dependent data (to be used in Program 2) and results are given.*

```
C     THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(50),XD(50),XDD(50),T(50)
      DATA M,K,C,X0,XD0,N,DELT/
     2 4.0,2500.0,400.0,0.1,-10.0,50,0.01/
C     END OF PROBLEM-DEPENDENT DATA

FREE VIBRATION ANALYSIS
OF A SINGLE DEGREE OF FREEDOM SYSTEM

DATA

M     = 0.40000000E+01
K     = 0.25000000E+04
C     = 0.40000000E+03
X0    = 0.10000000E+00
XD0   =-0.10000000E+02
N     =    50
DELT  = 0.99999998E-02


SYSTEM IS OVER DAMPED


RESULTS:

   I     TIME(I)          X(I)            XD(I)           XDD(I)
```

1	0.100000E-01	0.351455E-01	-0.390559E+01	0.368593E+03
2	0.200000E-01	0.990550E-02	-0.151007E+01	0.144816E+03
3	0.300000E-01	0.230870E-03	-0.569458E+00	0.568015E+02
4	0.400000E-01	-0.333730E-02	-0.201042E+00	0.221900E+02
5	0.500000E-01	-0.451879E-02	-0.576071E-01	0.858495E+01
6	0.600000E-01	-0.477582E-02	-0.257606E-02	0.324249E+01
7	0.700000E-01	-0.468266E-02	0.177700E-01	0.114966E+01
8	0.800000E-01	-0.446433E-02	0.245564E-01	0.334569E+00
9	0.900000E-01	-0.420854E-02	0.260878E-01	0.215607E-01
10	0.100000E+00	-0.394902E-02	0.256257E-01	-0.944359E-01
⋮				
46	0.460000E+00	-0.354993E-03	0.237800E-02	-0.159296E-01
47	0.470000E+00	-0.331992E-03	0.222393E-02	-0.148975E-01
48	0.480000E+00	-0.310481E-03	0.207983E-02	-0.139322E-01
49	0.490000E+00	-0.290364E-03	0.194507E-02	-0.130295E-01
50	0.500000E+00	-0.271551E-03	0.181905E-02	-0.121853E-01

(2.116) The equations for the natural frequencies of vibration were derived in Problem 2.35.

Operating speed of turbine is:

$$\omega_o = (2400)\frac{2\pi}{60} = 251.328 \text{ rad/sec}$$

Thus we need to satisfy:

$$\omega_n\Big|_{axial} = \left\{\frac{g\, l\, A\, E}{W\, a\,(l-a)}\right\}^{1/2} \geq \omega_o \qquad (E_1)$$

$$\omega_n\Big|_{transverse} = \left\{\frac{3\, E\, I\, l^3\, g}{W\, a^3\,(l-a)^3}\right\}^{1/2} \geq \omega_o \qquad (E_2)$$

$$\omega_n\Big|_{circumferential} = \left\{\frac{G\,J}{J_o}\left(\frac{1}{a}+\frac{1}{l-a}\right)\right\}^{1/2} \geq \omega_o \qquad (E_3)$$

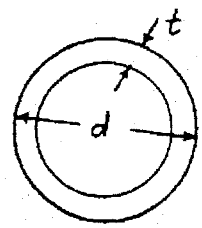
where $A = \frac{\pi d^2}{4}$, $W = 1000 \times 9.81 = 9810$ N,

$I = \frac{\pi d^4}{64}$, $J = \frac{\pi d^2}{32}$, $J_o = 500$ kg-m^2,

and $E = 207 \times 10^9$ N/m^2, $G = 79.3 \times 10^9$ N/m^2 (for steel).

The unknowns d, l and a can be determined to satisfy the inequalities (E_1), (E_2) and (E_3) using a trial and error procedure.

(2.117) From solution of problem 2.38, the requirements can be stated as:



$$\omega_n \Big|_{\text{pivot ends}} = \sqrt{\frac{12\, E I}{l^3 \left(\frac{W}{g} + m_{eff\,1} \right)}} \geq \omega_0 \qquad (E_1)$$

where $E = 30 \times 10^6$ psi and $I = \frac{\pi}{64} \left[d^4 - (d - 2t)^4 \right]$

$$\omega_n \Big|_{\text{fixed ends}} = \sqrt{\frac{48\, E I}{l^3 \left(\frac{W}{g} + m_{eff\,2} \right)}} \geq \omega_0 \qquad (E_2)$$

with

$m_{eff1} = (0.2357\ m)$, $m_{eff2} = (0.3714\ m)$,

m = mass of each column $= \frac{\pi}{4} \left[d^2 - (d-2t)^2 \right] \frac{l \rho}{g}$,

$\rho = 0.283$ lb/in^3, $g = 386.4$ in/sec^2,

l = length of column = 96 in.,

W = weight of floor = 4000 lb,

$$\underset{\sim}{W} = \text{weight of columns} = 4 \left\{ \frac{\pi}{4} \left[d^2 - (d-2t)^2 \right] l \rho \right\} \qquad (E_3)$$

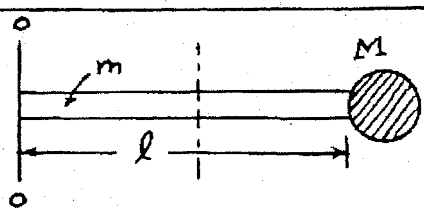
Frequency limit $= \omega_0 = 50 \times 2\pi = 314.16$ rad/sec.

Problem: Find d and t such that $\underset{\sim}{W}$ given by Eq. (E_3) is minimized while satisfying the inequalities (E_1) and (E_2).

This problem can be solved either by graphical optimization or by using a trial and error procedure.

(2.118)

$$J_0 = \frac{m l^2}{12} + \frac{m l^2}{4} + M l^2 = \frac{1}{3} m l^2 + M l^2 \qquad \text{--- } (E_1)$$



(i) Viscous damping:

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left(\frac{k_t}{\frac{1}{3} m l^2 + M l^2} \right)^{\frac{1}{2}} \qquad \text{--- } (E_2)$$

$$(c_t)_{cri} = 2\, J_0\, \omega_n = 2 \sqrt{J_0\, k_t} \qquad \text{--- } (E_3)$$

For critical damping, Eq. (2.80) gives

$$\theta(t) = \left\{ \theta_0 + \left(\dot{\theta}_0 + \omega_n \theta_0 \right) t \right\} e^{-\omega_n t} \qquad \text{--- } (E_4)$$

For $\theta_0 = 75^\circ = 1.309$ rad and $\dot{\theta}_0 = 0$,

$$\theta(t) = \left(1.309 + 1.309\,\omega_n t\right) e^{-\omega_n t} \qquad \text{--- } (E_5)$$

For $\theta = 5^\circ = 0.08727$ rad, Eq. (E_5) becomes

$$0.08727 = 1.309\,(1 + \omega_n t)\, e^{-\omega_n t} \qquad \text{--- } (E_6)$$

Let time to return = 2 sec. Then Eq. (E_6) gives

$$0.08727 = 1.309\,(1 + 2\omega_n)\, e^{-2\omega_n} \qquad \text{--- } (E_7)$$

Solve (E_7) by trial and error to find ω_n. Then choose the values of m, M and k_t to get the desired value of ω_n. Find the damping constant $(c_t)_{cri}$ using Eq. (E_3).

(ii) Coulomb damping:

(a) Follow the procedure of part(i) to find the value of ω_n.

(b) Derive expression for the equivalent torsional viscous damping constant $(c_t)_{eq}$ for Coulomb damping. This expression, for small amounts of damping, is

$$(c_t)_{eq} = \left\{ 4\, T_d / \pi\, \omega_n\, \Theta \right\} \qquad \text{--- } (E_8)$$

where T_d = friction (damping) torque, and Θ = amplitude of angular oscillations.

(c) If $(c_t)_{eq}$ is to be equal to $(c_t)_{cri} = 2\sqrt{J_0\, k_t}$, we find

$$T_d = \frac{\pi\, \omega\, \Theta}{4} \left(2\sqrt{J_0\, k_t} \right) \qquad \text{--- } (E_9)$$

2.119 Let x = vertical displacement of the mass (lunar excursion module), x_s = resulting deflection of each inclined leg (spring). From equivalence of potential energy, we find:
k_{eq_1} = stiffness of each leg in vertical direction $= k \cos^2 \alpha$
Hence for the four legs, the equivalent stiffness in vertical direction is:

$$k_{eq} = 4\, k \cos^2 \alpha$$

Similarly, the equivalent damping coefficient of the four legs in vertical direction is:

$$c_{eq} = 4\, c \cos^2 \alpha$$

where c = damping constant of each leg (in axial motion). Modeling the system as a single degree of freedom system, the equation of motion is:

$$m_{eq}\, \ddot{x} + c_{eq}\, \dot{x} + k_{eq}\, x = 0$$

and the damped period of vibration is:

$$\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{2\pi}{\sqrt{\frac{k_{eq}}{m_{eq}}}\sqrt{1-\left(\frac{c_{eq}^2}{4\, k_{eq}\, m_{eq}}\right)}}$$

Using m_{eq} = 2000 kg, $k_{eq} = 4\, k \cos^2 \alpha$, $c_{eq} = 4\, c \cos^2 \alpha$, and $\alpha = 20°$, the values of k and c can be determined (by trial and error) so as to achieve a value of τ_d between 1 s and 2 s. Once k and c are known, the spring (helical) and damper (viscous) can be designed suitably.

2.120 Assume no damping. Neglect masses of telescoping boom and strut. Find stiffness of telescoping boom in vertical direction (see Example 2.2). Find the equivalent stiffness of telescoping boom together with the strut in vertical direction. Model the system as a single degree of freedom system with natural time period:

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m_{eq}}{k_{eq}}}$$

Using τ_n = 1 s and $m_{eq} = \left(\frac{W_c + W_f}{g}\right) = \frac{300}{386.4}$, determine the axial stiffness of the strut (k_s). Once k_s is known, the cross section of the strut (A_s) can be found from:

$$k_s = \frac{A_s\, E_s}{\ell_s}$$

with $E_s = 30\,(10^6)$ psi and ℓ_s = length of strut (known).

Chapter 3

Harmonically Excited Vibration

3.1 (a) $\delta = \frac{W}{k} = \frac{50}{4000} = 0.0125 \text{ m}$

(b) $\delta_{st} = \frac{F_0}{k} = \frac{60}{4000} = 0.015 \text{ m}$

(c) $\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{4000 \times 9.81}{50}\right)^{1/2} = 28.0143 \text{ rad/sec}$

$\omega = 6 \text{ Hz} = 37.6992 \text{ rad/sec}$

$$X = \delta_{st} \left| \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \right| = 0.015 \left| \frac{1}{1-\left(\frac{37.6992}{28.0143}\right)^2} \right| = 0.0185 \text{ m}$$

3.2 $\tau_b = \frac{2\pi}{\omega_n - \omega} = \frac{2\pi}{2\pi\,(40.0-39.8)} = 5 \text{ sec}$

3.3 $\delta_{st} = \frac{F_0}{k} = \frac{25}{2000} = 0.0125 \text{ m}$

steady state solution at resonance $= x(t) = \frac{\delta_{st} \cdot \omega_n t}{2} \sin \omega_n t$

$= 0.00625\ \omega_n t \sin \omega_n t \quad \text{m}$

(a) At end of $\frac{1}{4}$ cycle, $\omega_n t = \frac{\pi}{2}$ and $x(t) = 0.00625\left(\frac{\pi}{2}\right) \sin \frac{\pi}{2} = 0.009817 \text{ m}$

(b) At end of $2\frac{1}{2}$ cycles, $\omega_n t = 5\pi$ and $x(t) = 0.00625\,(5\pi) \sin 5\pi = 0$

(c) At end of $5\frac{3}{4}$ cycles, $\omega_n t = 11\frac{1}{2}\pi$ and

$x(t) = 0.00625\left(\frac{23}{2}\pi\right) \sin \frac{23}{2}\pi = -0.2258 \text{ m}$

3.4 $\delta_{st} = \frac{F_0}{k} = \frac{100}{4000} = 0.025 \text{ m}$

$$X = \delta_{st} \cdot \frac{1}{\left|1-\left(\frac{\omega}{\omega_n}\right)^2\right|}, \quad \left|1-\left(\frac{\omega}{\omega_n}\right)^2\right| = \frac{\delta_{st}}{X} = \frac{0.025}{20 \times 10^{-3}} = 1.25$$

$\frac{\omega}{\omega_n} = \sqrt{1.25+1} = 1.5$

$\omega_n = \omega/1.5 = 5(2\pi)/1.5 = 20.944 \text{ rad/sec}$

$m = k/\omega_n^2 = 4000/(20.944)^2 = 9.1189 \text{ kg}$

3.5 $\omega_n = \sqrt{k/m} = \sqrt{5000/10} = 22.3607 \text{ rad/sec}$

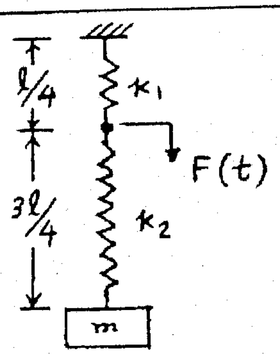
$\delta_{st} = F_0/k = 250/5000 = 0.05 \text{ m}$

$$X = \delta_{st} \left\{ \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \right\}$$

i.e., $$\omega = \omega_n \left(1 - \frac{\delta_{st}}{X}\right)^{\frac{1}{2}} = 22.3607 \left[1 - \frac{0.05}{0.10}\right]^{\frac{1}{2}}$$

$$= 15.8114 \text{ rad/sec}$$

(3.6)



$$k_1 = 4k \;; \quad \frac{1}{4k} + \frac{1}{k_2} = \frac{1}{k}, \quad k_2 = \frac{4}{3} k$$

Force transmitted to the mass through k_2:

$$\tilde{F}(t) = \frac{k_2}{k_1 + k_2} F(t) = \frac{k_1 k_2}{k_1 + k_2} \left(\frac{F_0}{k_1}\right) \cos \omega t$$

$$= k \; \delta_{st} \cos \omega t \quad \text{where} \quad \delta_{st} = \frac{F_0}{k_1}$$

Steady state response of m:

$$x(t) = \frac{\tilde{F}_0}{k \left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}} \cos \omega t$$

$$= \left\{ \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right\} \cos \omega t \quad \text{with} \quad \tilde{F}_0 = k \cdot \delta_{st}.$$

(3.7) Equivalent stiffness of wing (beam) at location of engine:

$$k = \frac{\text{force}}{\text{deflection}} = \frac{3\,E\,I}{\ell^3} = \frac{3\,E\,(\frac{1}{12}\,b\,a^3)}{\ell^3} = \frac{E\,b\,a^3}{4\,\ell^3}$$

Magnitude of unbalanced force: $$= m\,r\,\omega^2 = m\,r \left(\frac{2\,\pi\,N}{60}\right)^2 = \frac{m\,r\,\pi^2\,N^2}{900}$$

Equivalent mass of wing at location of engine: $$M = \frac{33}{140}\,m_w = \frac{33}{140}\,(a\,b\,\ell\,\rho)$$

Equation of motion: $M\,\ddot{x} + k\,x = m\,r\,\omega^2 \sin \omega\,t$

Maximum steady state displacement of wing at location of engine:

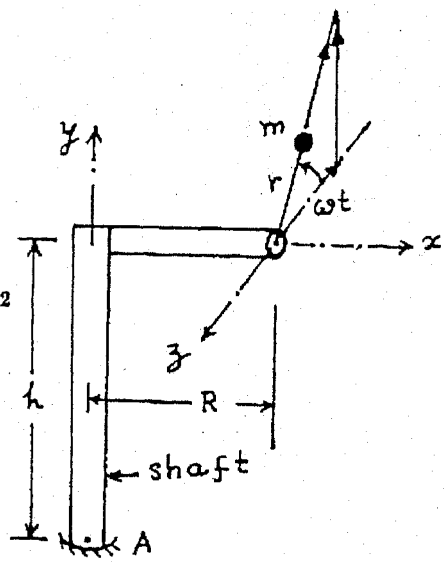
$$X = \left| \frac{m\,r\,\omega^2}{k - M\,\omega^2} \right| = \left| \frac{\left(\frac{m\,r\,\pi^2\,N^2}{900}\right)}{\left\{\frac{E\,b\,a^3}{4\,\ell^3} - \frac{33}{140}\,a\,b\,\ell\,\rho \left(\frac{2\,\pi\,N}{60}\right)^2\right\}} \right|$$

$$= \left| \frac{m\,r\,\ell^3\,N^2}{22.7973\,E\,b\,a^3 - 0.2357\,\rho\,a\,b\,\ell^4\,N^2} \right|$$

3.8 Rotating unbalanced force, $m\,r\,\omega^2$, can be resolved into two components as:

$$F_y = m\,r\,\omega^2 \sin \omega t \quad \text{(parallel y—axis)}$$
$$F_z = m\,r\,\omega^2 \cos \omega t \quad \text{(parallel z—axis)}$$

Maximum bending stress at A:



$$\sigma_b = \frac{1}{I_z}\,|M_z|\,\frac{d_o}{2} = \frac{m\,r\,\omega^2\,R\left(\dfrac{d_o}{2}\right)}{\dfrac{\pi}{64}\left(d_o^4 - d_i^4\right)}$$

$$= \frac{(0.1)\,(0.1)\,(31.416^2)\,(0.5)\,(\frac{0.1}{2})}{\dfrac{\pi}{64}\,(0.1^4 - 0.08^4)} = 8.5124\,(10^4)\ \text{N/m}^2$$

Maximum torsional stress at A:

$$\sigma_t = \frac{1}{J_y}\,|M_y|\left(\frac{d_o}{2}\right) = \frac{m\,r\,\omega^2\,R\left(\dfrac{d_o}{2}\right)}{\dfrac{\pi}{32}\left(d_o^4 - d_i^4\right)}$$

$$= 4.2562\,(10^4)\ \text{N/m}^2$$

3.9 Total stiffness with steel specimen:

$$k_{eq} = k_1 + k_2 = 10{,}217.0296 + 750{,}000.0 = 760{,}217.0296\ \text{lb/in}$$

Force in specimen due to magnets (static) due to elongation $X = k_2\,X$.
Force in specimen due to a.c. current in magnets (dynamic) due to elongation $X = k_{eq}\,X - m\,\omega^2\,X$.

$$\text{Ratio of stresses} = \left|\frac{k_2\,X}{k_{eq}\,X - m\,\omega^2\,X}\right| = \frac{1}{2} \quad \text{i.e.,} \quad \left|\frac{k_2}{k_{eq} - m\,\omega^2}\right| = \frac{1}{2}.\text{sp}$$

$$\text{i.e.,} \quad \left|\frac{750{,}000.0}{760{,}217.0296 - \left(\dfrac{40}{386.4}\right)\omega^2}\right| = \frac{1}{2}$$

Squaring both sides of this equation and rearranging gives:

$$107.1225\,\omega^4 - 15.7365\,(10^8)\,\omega^2 - 167.207\,(10^{14}) = 0$$
$$\text{or} \quad \omega^2 = 0.218378\,(10^8) \quad \text{(positive value)}$$
$$\omega = 4673.0935\ \text{rad/sec} = 743.7442\ \text{Hz}$$

3.10 Equation of motion: $m_{eq}\,\ddot{x} + k_{eq}\,x = F(t)$
where m_{eq} = mass of valve and valve rod plus mass of spring at end = $(20 + (15/3))/386.4 = 0.0647$ lb—sec^2/in.
$k_{eq} = 400$ lb/in, $F(t) = A\ p(t) = A\,p_0 \sin \omega t = 100\,(10) \sin \omega t = 1000 \sin 8t$ lb.
Response of valve (steady state) $= x_p(t) = X \sin 8t$ in where

$$X = \frac{1000}{\quad} = 2.5261\ \text{in}$$

3.11 (a) Equation of motion:

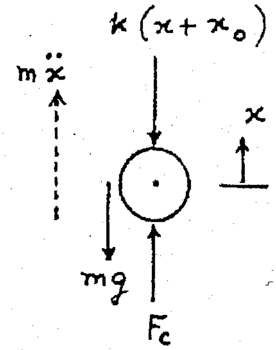
$$m_0\, \ddot{x} + k\,(x + x_0) + m_0\, g = F_c;\ \ \ddot{x} > 0 \tag{1}$$

where F_c = force exerted on the follower by the cam, m_0 = mass of follower plus one third the mass of the spring, and x_0 = initial displacement of the spring.

(b) Force exerted on the follower by the cam:

$$F_c = m_0\, \ddot{x} + k\,(x + x_0) + m_0\, g \tag{2}$$

with $x = e \cos \omega t$.

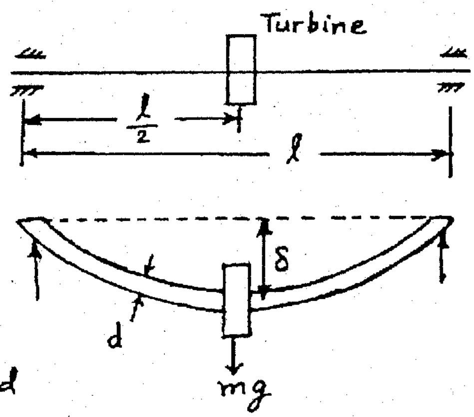


(c) Condition under which follower loses contact with the cam is when F_c is zero and $\ddot{x}$ is negative. Equation (1) can be used to state this condition as:

$$k\,(x + x_0) + m_0\, g \geq |\, m_0\, \ddot{x}\,| \tag{3}$$

3.12 δ_{st} = static radial displacement of shaft under weight of turbine

δ = radial deflection of shaft during rotation



$k = \dfrac{48\,E\,I}{l^3}$ = stiffness of centrally loaded simply supported beam

$$m\,\delta\,\omega^2 = k\,(\delta - \delta_{st}) \quad \text{or} \quad m\,\omega^2 = k - k\cdot\frac{\delta_{st}}{\delta}$$

or

$$\frac{\delta}{\delta_{st}} = \frac{k}{k - m\,\omega^2} \tag{E$_1$}$$

Critical speed is $\omega_{cri} = \sqrt{\dfrac{k}{m}}$ (E$_2$)

If critical speed = $\frac{1}{5}$th of operating speed,

$$\sqrt{\frac{k}{m}} = \frac{1}{5}\,\omega \tag{E$_3$}$$

Here $m = 500/386.4 = 1.2940$ lb-s^2/in

and $\omega = 3000 \times 2\pi/60 = 314.16$ rad/sec

For solid shaft (steel) of diameter d and length l, Eq. (E$_3$) gives

$$\frac{48\,E\,I}{m\,\ell^3} = \frac{\omega^2}{25} \quad \text{with } E = 30 \times 10^6 \text{ psi and } I = \frac{\pi d^4}{64}$$

i.e., $$\frac{\ell^3}{d^4} = 13836.8 \qquad (E_4)$$

Let $\ell = 30\,d$ in Eq. (E_4):

$$d = \frac{27000}{13836.8} = 1.9513 \text{ inch} \quad \text{and hence } \ell = 58.5395 \text{ inch.}$$

3.13 $I = \frac{\pi}{64}\left(d_o^4 - d_i^4\right) = \frac{\pi}{64}\left(4^4 - 3.5^4\right) = 5.2002 \text{ in}^4$

$$k = \frac{48\,E\,I}{\ell^3} = \frac{48\,(30 \times 10^6)\,(5.2002)}{(100)^3} = 7488.288 \text{ lb/in}$$

$m = 500/386.4 = 1.2940 \text{ lb} - \text{s}^2/\text{in}$

$\omega_n = \sqrt{k/m} = \sqrt{7488.288/1.2940} = 76.0719 \text{ rad/sec}$

Eccentricity $= r = 2$ in, eccentric mass $= m_o = \frac{0.5}{386.4} \frac{\text{lb}-\text{s}^2}{\text{in}}$

Radial force due to eccentric mass at resonance

$$= F_0 = m_o\, r\, \omega^2 = \left(\frac{5}{386.4}\right)(2)(76.0719)^2 = 149.7654 \text{ lb}$$

Let $x(t)$ = radial displacement of turbine.

At resonance, Eq. (3.15) gives, for $x_o = \dot{x}_o = 0$,

$$x(t) = \frac{1}{2}\,\delta_{st}\,\omega_n t \sin \omega_n t$$

where $$\delta_{st} = \frac{F_0}{k} = \frac{149.7654}{7488.288} = 0.02 \text{ in}$$

To activate the limit switch, $x(t) = 0.5$ in. and hence

$$0.5 = \frac{1}{2}(0.02)(76.0719)\,t \sin 76.0719\,t$$

i.e., $$t \sin 76.0719\,t = 0.6573 \qquad (E_1)$$

Eq. (E_1) is solved by trial and error (assuming values of $t = 1.0, 0.9, 0.8, 0.7$, etc.) as

$$t \simeq 0.6760 \text{ sec.}$$

3.14 Tip load $= 0.1$ lb, tip mass $= m_o = \frac{0.1}{386.4} = 2.588 \times 10^{-4}$ lb $-$ s^2/in

$I = \frac{1}{12}(0.2)(0.05)^3 = 2.0833 \times 10^{-6} \text{ in}^4$

$$k = \frac{3EI}{l^3} = \frac{3(30\times 10^6)(2.0833\times 10^{-6})}{(10)^3} = 0.1875 \ \text{lb/in}$$

$$m = \text{mass of beam} = \frac{0.283}{386.4}(10 \times 0.2 \times 0.05) = 7.324 \times 10^{-5} \ \text{lb-s}^2/\text{in}$$

$$\omega_n = \left(\frac{k}{m_0 + 0.23\,m}\right)^{\frac{1}{2}} = \left[\frac{0.1875}{(2.588 + 0.7324)\,10^{-4}}\right]^{\frac{1}{2}} = 5.6469 \ \frac{\text{rad}}{\text{sec}}$$

Eq. (3.68) gives

$$\frac{X}{Y} = \left\{\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right\}^{\frac{1}{2}}$$

i.e.,

$$\frac{2.5}{0.05} = 50 = \left\{\frac{1 + (2 \times 0.01\, r)^2}{(1-r^2)^2 + (2 \times 0.01\, r)^2}\right\}^{\frac{1}{2}}$$

i.e.

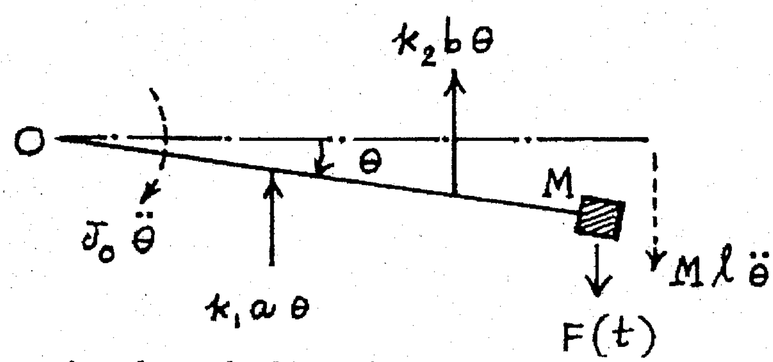
$$r^4 - 1.9996\, r^2 + 0.9996 = 0$$

i.e.

$$r = \frac{\omega}{\omega_n} = 0.9999$$

$$\therefore \quad \omega = 0.9999\, \omega_n = 5.6463 \ \text{rad/sec.}$$

3.15



Equation of motion for rotational motion about the hinge O:

$$(J_0 + M\,\ell^2)\,\ddot{\theta} + (k_1\, a^2 + k_2\, b^2)\,\theta = F(t)\,\ell = F_0\,\ell \sin \omega t \tag{1}$$

Steady state response (using Eqs. (3.3) and (3.6)):

$$\theta_p(t) = \Theta \sin \omega t \tag{2}$$

$$\text{where} \quad \Theta = \frac{F_0\,\ell}{(k_1\, a^2 + k_2\, b^2) - (J_0 + M\,\ell^2)\,\omega^2} \tag{3}$$

$$\text{and} \quad J_0 = \frac{m\,\ell^2}{12} + m\left(\frac{\ell}{2}\right)^2 = \frac{1}{3}\, m\,\ell^2 \tag{4}$$

For given data, $J_0 = \frac{1}{3}(10)(1^2) = 3.3333$ kg–m^2, $\omega = \frac{1000\,(2\,\pi)}{60} = 104.72$ rad/sec, and

$$\Theta = \frac{500\ (1)}{5000\ (0.25^2 + 0.5^2) - (3.3333 + 50\ (1^2))\ (104.72^2)} = -\ 8.5718\ (10^{-4})\ \text{rad}$$

3.16 Equation of motion for rotation about O:

$$J_0\ \ddot{\theta} = -\ k\ \frac{\theta\ \ell}{4}\ \frac{\ell}{4} - k\ \frac{\theta\ 3\ \ell}{4}\ \frac{3\ \ell}{4} + M_0\ \cos\ \omega\ t$$

$$\text{i.e.,}\quad J_0\ \ddot{\theta} + \left(\frac{5}{8}\ k\ \ell^2\right)\ \theta = M_0\ \cos\ \omega\ t$$

where $J_0 = \frac{1}{12}\ m\ \ell^2 + m\ (\frac{\ell}{4})^2 = \frac{7}{48}\ m\ \ell^2 = \frac{7}{48}\ (10)\ (1^2) = 1.4583\ \text{kg--m}^2$

and $\omega = 1000$ rpm $= 104.72$ rad/sec. Steady state solution is:

$$\theta_p(t) = \Theta\ \cos\ \omega\ t$$

where

$$\Theta = \frac{M_0}{\frac{5}{8}\ k\ \ell^2 - J_0\ \omega^2} = \frac{100}{5000\ (\frac{5}{8})\ (1^2) - 1.4583\ (104.72^2)} = -\ 0.007772\ \text{rad}$$

3.17 $m = \frac{500}{386.4}$ lb–sec^2/in, $F(t) = 200\ \sin\ 100\ \pi\ t$ lb. Let $X_{max} = 0.05$ in < 0.1 in (maximum permissible value). From Eq. (3.33),

$$X_{max} = \delta_{st}\ \frac{1}{2\ \varsigma\ \sqrt{1 - \varsigma^2}} = 0.05 \tag{1}$$

Let $\varsigma = 0.01$. Then $\delta_{st} = \frac{F_0}{k_{eq}} = \frac{200}{k_{eq}}$ and Eq. (1) gives

$$k_{eq} = \frac{200}{2\ (0.01)\ \sqrt{1 - 0.0001}\ (0.05)} = 20.0020\ (10^4)\ \text{lb/in}$$

Since shock mounts are in parallel, stiffness of each mount $= k = \frac{k_{eq}}{3} = 6.6673\ (10^4)$ lb/in.

$$\varsigma = \frac{c_{eq}}{c_c} = \frac{c_{eq}}{\sqrt{2\ k_{eq}\ m}}$$

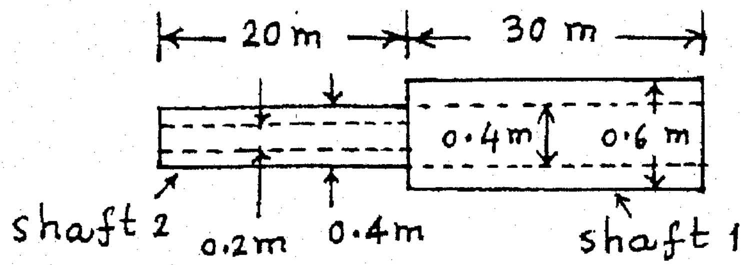
$$\text{or}\quad c_{eq} = \varsigma\ \sqrt{2\ k_{eq}\ m} = 0.01\ \sqrt{2\ (20.0020\ (10^4))\ (\frac{500}{386.4})} = 7.1948\ \text{lb--sec/in}$$

and hence $c = \frac{c_{eq}}{3} = 2.3983$ lb-sec/in

3.18 Equation of motion for torsional system:

$$J_0\ \ddot{\theta} + c_t\ (\dot{\theta} - \dot{\alpha}) + k_t\ (\theta - \alpha) = 0 \tag{1}$$

where $\theta =$ angular displacement of shaft and $\alpha =$ angular displacement of base of shaft $= \alpha_0\ \sin\ \omega\ t$. Steady state response of propeller (Eq. (3.67)):



$$\theta_p(t) = \Theta \sin(\omega t - \phi) \tag{2}$$

$$\text{where } \Theta = \alpha_0 \left\{ \frac{k_t^2 + (c_t\,\omega)^2}{(k_t - J_0\,\omega^2)^2 - (c_t\,\omega)^2} \right\}^{\frac{1}{2}} \tag{3}$$

$$\text{and } \phi = \tan^{-1} \left\{ \frac{J_0\ c_t\ \omega^3}{k_t\,(k_t - J_0\,\omega^2) + (c_t\,\omega)^2} \right\} \tag{4}$$

Here $J_0 = 10^4$ kg–m^2, $\varsigma_t = 0.1$, and $\omega = 314.16$ rad/sec. Torsional stiffnesses of shafts:

$$(k_t)_1 = \frac{G_1\ J_1}{\ell_1} = \frac{(80\,(10^9))\left[\frac{\pi}{32}\,(0.6^4 - 0.4^4)\right]}{30} = 27.2272\,(10^6)\ \text{N–m/rad}$$

$$(k_t)_2 = \frac{G_2\ J_2}{\ell_2} = \frac{(80\,(10^9))\left[\frac{\pi}{32}\,(0.4^4 - 0.2^4)\right]}{20} = 9.4248\,(10^6)\ \text{N–m/rad}$$

Series springs give:

$$k_t = \frac{(k_t)_1\ (k_t)_2}{(k_t)_1 + (k_t)_2} = \frac{(27.2272\,(10^6))\,(9.4248\,(10^6))}{27.2272\,(10^6) + 9.4248\,(10^6)} = 7.0013\,(10^6)\ \text{N–m/rad}$$

$$c_t = \varsigma\,(2\sqrt{J_0\ k_t}) = 0.1\,(2)\sqrt{(10^4)\,(7.0013\,(10^6))} = 52{,}919.8624\ \text{N–m–s/rad}$$

From Eq. (3),

$$\Theta = 0.05 \left[\frac{(7.0013\,(10^6))^2 + \left\{5.2920\,(10^4)\,(314.16^2)\right\}^2}{\left\{7.0013\,(10^6) - (10^4)\,(314.16^2)\right\}^2 + \left\{5.2920\,(10^4)\,(314.16)\right\}^2} \right]^{\frac{1}{2}}$$

$$= 9.2028\,(10^{-4})\ \text{rad}$$

$$\phi = \tan^{-1} \left\{ \frac{(10^4)\,(5.2920\,(10^4))\,(314.16^3)}{7.0013\,(10^6)\left[7.0013\,(10^6) - (10^4)\,(314.16^2)\right] + (5.2920\,(10^4)\,(314.16))^2} \right\}$$

$$= \tan^{-1}(59.3664) = 89.0350^\circ = 1.5540\ \text{rad}$$

3.19 $X = \dfrac{\delta_{st}}{\left\{(1-r^2)^2 + (2\zeta r)^2\right\}^{1/2}}$

For maximum X, $\dfrac{dX}{dr} = -\delta_{st} \cdot \dfrac{1}{2} \dfrac{1}{\left\{(1-r^2)^2 + (2\zeta r)^2\right\}^{3/2}} \cdot \left\{2(1-r^2)(-2r) + 2(2\zeta r)(2\zeta)\right\}$
$= 0$

i.e., $-4r(1-r^2) + 8r\zeta^2 = 0$

i.e., $r = \sqrt{1-2\zeta^2}$

$$X\Big|_{at\ r = \sqrt{1-2\zeta^2}} = \frac{\delta_{st}}{\left[\left\{1-(1-2\zeta^2)\right\}^2 + \left(2\zeta\sqrt{1-2\zeta^2}\right)^2\right]^{1/2}} = \frac{\delta_{st}}{2\zeta\sqrt{1-\zeta^2}}$$

$$\therefore \left(\frac{X}{\delta_{st}}\right)_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

3.20 Under a d.c. current (I) through the coil, core rotates by angle θ. Torque developed due to I balances the restoring torque of spring: $a\,I = k_t\,\theta$ where a is a constant and k_t is the torsional spring constant. Under an a.c. current I(t), torque developed is $T(t) = a\,I(t)$ and the equation of motion is:

$$J_0\,\ddot{\theta} + c_t\,\dot{\theta} + k_t\,\theta = T(t) = a\,I(t) = a\,I_0 \cos\omega t \tag{1}$$

Steady state angular displacement of core:

$$\theta_p(t) = \Theta \cos(\omega t - \phi).sp \tag{2}$$

$$\text{where}\quad \Theta = \frac{a\,I_0}{\left\{(k_t - J_0\,\omega^2)^2 + (c_t\,\omega)^2\right\}^{\frac{1}{2}}} = \frac{\left(\dfrac{a\,I_0}{k_t}\right)}{\left[\left\{1-(\dfrac{\omega}{\omega_n})^2\right\}^2 + \left\{2\,\zeta\,\dfrac{\omega}{\omega_n}\right\}^2\right]^{\frac{1}{2}}} \tag{3}$$

When $\omega = 0$ (d.c. current) and $I_0 = 1$ ampere, Eq. (1) gives

$$\Theta_{dc} = \left(\frac{a}{k_t}\right) = 1 \text{ (reading corresponding } \Theta_{dc})$$

and hence $a = k_t = 62.5$.

When $\omega = 50$ Hz $= 314.16$ rad/sec and $I_0 = 5$ amperes, Eq. (3) gives:

$$\Theta_{ac} = \frac{\left(\dfrac{a\,(5)}{k_t}\right)}{\left[\left\{1-\left(\dfrac{314.16}{250}\right)^2\right\}^2 + \left\{2\,(1)\left(\dfrac{314.16}{250}\right)\right\}^2\right]^{\frac{1}{2}}} = 1.9386 \text{ amperes}$$

where $J_0 = 0.001$ N–m^2, $k_t = 62.5$ N-m/rad, $c_t = 0.5$ N-m-s/rad, and $\omega_n = \sqrt{\dfrac{k_t}{J_0}} = \sqrt{\dfrac{62.5}{0.001}} = 250$ rad/s. The steady state value of current indicated by ammeter = 1.9386 amperes (this shows that the ammeter is not accurate).

3.21 Eq. (3.34):
$$\frac{X_{res}}{\delta_{st}} = \left.\frac{X}{\delta_{st}}\right|_{\omega=\omega_n} = \frac{1}{2\zeta}$$

i.e.,
$$\delta_{st} = 2\,\zeta\left(\frac{20}{1000}\right) = 0.04\,\zeta \qquad (E_1)$$

Eq. (3.30):
$$\frac{X}{\delta_{st}} = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \; ; \quad r = 0.75 = \frac{\omega}{\omega_n}$$

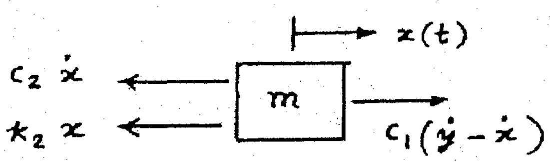
i.e.,
$$\frac{0.01}{\delta_{st}} = \frac{1}{\sqrt{(1-0.75^2)^2+(2\,\zeta\times 0.75)^2}} \qquad (E_2)$$

Eqs. (E_1) and (E_2) give
$$\frac{0.01}{0.04\,\zeta} = \frac{1}{\sqrt{0.1914+2.25\,\zeta^2}}$$

i.e., $\quad 0.1914 + 2.25\,\zeta^2 = 16\,\zeta^2$

i.e., $\quad \zeta = 0.1180$

3.22 (a) Equation of motion of mass:



$$m\ddot{x} = c_1(\dot{y}-\dot{x}) - c_2\dot{x} - k_2 x$$

i.e.,
$$m\ddot{x} + (c_1+c_2)\,\dot{x} + k_2 x = c_1\,\dot{y} = -c_1\omega Y \sin\omega t$$

(b)
$$x_p(t) = \frac{-(c_1\omega Y/k_2)}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\,\sin(\omega t-\phi)$$

where $r = \omega/\omega_n$, $\zeta = (c_1+c_2)\,\omega/(2\,r\,k)$ and $\phi = \tan^{-1}\left(\dfrac{2\zeta r}{1-r^2}\right)$.

(c) steady-state force transmitted to point P:
$$= k_2\,x_p + c_2\,\dot{x}_p$$
$$= \frac{-(c_1\omega Y)}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\left\{\sin(\omega t-\phi) + \frac{c_2\,\omega}{k_2}\cos(\omega t-\phi)\right\}$$

(3.23) Eq. (3.34) gives $\left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta}$

If $X = \frac{1}{\sqrt{2}} X_{max} = \frac{1}{\sqrt{2}} X\Big|_{\omega=\omega_n}$, Eq. (3.30) gives

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{2\zeta} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Squaring and rearranging

$$8\zeta^2 = (1-r^2)^2 + 4\zeta^2 r^2 = 1 - 2r^2 + r^4 + 4r^2\zeta^2$$

$$r^4 + r^2(4\zeta^2 - 2) + (1 - 8\zeta^2) = 0$$

$$r^2 = 1 - 2\zeta^2 \pm 2\zeta\sqrt{1+\zeta^2}$$

Neglecting terms involving ζ^2,

$$r^2 = \frac{\omega^2}{\omega_n^2} = 1 \pm 2\zeta$$

Let $\omega = \omega_1$ when $r^2 = 1 - 2\zeta$ and $\omega = \omega_2$ when $r^2 = 1 + 2\zeta$

$$\frac{\omega_2^2 - \omega_1^2}{\omega_n^2} = \frac{(\omega_2 + \omega_1)(\omega_2 - \omega_1)}{\left(\frac{\omega_2 + \omega_1}{2}\right)^2} = 4\zeta$$

$$\therefore \quad \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1} = \zeta$$

(3.24) $$k_t = \frac{\pi G}{32\, l} d^4 = \frac{\pi (79.3 \times 10^9)}{32 (1)} \left(\frac{4}{100}\right)^4 = 19930.31 \text{ N-m/rad}$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{19930.31}{10}} = 44.6434 \text{ rad/sec}$$

$$\theta_{st} = M_{t0}/k_t = 1000/19930.31 = 0.0502 \text{ rad}$$

$$\zeta_t = \frac{c_t}{2 J_0 \omega_n} = \frac{300}{2(10)(44.6434)} = 0.336$$

(a) Eq. (3.30), when written for a torsional system, gives

$$\frac{\Theta}{\theta_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

i.e., $$\frac{(2/57.2956)}{0.0502} = \frac{1}{\sqrt{(1-r^2)^2 + (2 \times 0.336\, r)^2}}$$

i.e., $$r^4 - 1.5484\, r^2 - 1.0679 = 0$$

i.e., $$r^2 = 2.0655,\ -0.5171$$

$$\therefore \quad \omega = r\,\omega_n = \sqrt{2.0655}\,(44.6434) = 64.16 \text{ rad/sec}$$

(b) Maximum torque transmitted to the support:

$$M_t(t) = k_t\,\theta(t) + c_t\,\dot{\theta}(t)$$

$$= k_t\,\Theta\cos(\omega t-\phi) - c_t\,\Theta\,\omega\sin(\omega t-\phi)$$

$$(M_t)_{max} = \sqrt{(k_t\,\Theta)^2 + (c_t\,\Theta\,\omega)^2}$$

$$= \sqrt{\left\{19930.31\left(\frac{2}{57.2956}\right)\right\}^2 + \left\{300\left(\frac{2}{57.2956}\right)(64.16)\right\}^2}$$

$$= 967.2 \text{ N-m}$$

3.25 Complete solution is $x(t) = X_o\,e^{-\zeta\omega_n t}\cos(\omega_d t+\phi_o) + X\cos(\omega t-\phi)$

$\omega = 2\pi(3.5) = 21.9912$ rad/sec, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{10}} = 15.8114$ rad/sec

$\delta_{st} = \frac{F_o}{k} = \frac{180}{2500} = 0.072$ m

$\zeta = \frac{c}{2m\omega_n} = \frac{45}{2(10)(15.8114)} = 0.1423$, $r = \frac{\omega}{\omega_n} = \frac{21.9912}{15.8114} = 1.3908$

$\zeta\omega_n = 2.25$, $\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 15.6505$

$$X = \frac{\delta_{st}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{0.072}{\left[(1-1.3908^2)^2 + (2\times0.1423\times1.3908)^2\right]^{1/2}}$$

$= 0.07095$ m

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{0.3958}{-0.9343}\right) = -22.9591^\circ$$

$$x(t) = X_o\,e^{-2.25t}\cos(15.6505t+\phi_o) + 0.07095\cos(21.9912t + 22.9591^\circ)$$

$$\dot{x}(t) = -2.25\,X_o\,e^{-2.25t}\cos(15.6505t+\phi_o) - 15.6505\,X_o\,e^{-2.25t}\sin(15.6505t+\phi_o)$$
$$- 21.9912(0.07095)\sin(21.9912t + 22.9591^\circ)$$

$$x(0) = 0.015 = X\cos\phi_o + 0.07095\cos 22.9591^\circ$$

$$X\cos\phi_o = -0.05033 \quad ---- (E_1)$$

$$\dot{x}(0) = 5 = -2.25\,X_o\cos\phi_o - 15.6505\,X_o\sin\phi_o - 1.5603\sin 22.9591^\circ$$

$$X\sin\phi_o = \frac{-0.6086 - 2.25\,X_o\cos\phi_o - 5}{15.6505} = -0.3511 \quad ---- (E_2)$$

Eqs. (E_1) and (E_2) give

$$X_o = \left\{(-0.05033)^2 + (-0.3511)^2\right\}^{1/2} = 0.3547$$

$$\phi_o = \tan^{-1}\left(\frac{0.3511}{0.05033}\right) = \tan^{-1}(6.9760) = 81.8423^\circ$$

3.26 (a) Eq. (3.38) gives $\frac{1}{2\zeta} \approx \left(\frac{X}{\delta_{st}}\right)_{max} = \frac{0.2}{0.1} = 2$

$\therefore \zeta = 0.25$

(b) Eqs. (3.42) yield

$$\left(\frac{\omega_1}{\omega_n}\right)^2 \approx 1 - 2\zeta = 0.5 \;, \quad \omega_1 = \omega_n \sqrt{0.5} = (5 \times 2\pi)\sqrt{0.5} = 22.2145 \text{ rad/sec}$$

$$\left(\frac{\omega_2}{\omega_n}\right)^2 \approx 1 + 2\zeta = 1.5 \;, \quad \omega_2 = \omega_n \sqrt{1.5} = (5 \times 2\pi)\sqrt{1.5} = 38.4766 \text{ rad/sec}$$

3.27 Amplitude of vibration under base excitation:

$$X = Y \left\{ \frac{\sqrt{k^2 + (c\,\omega)^2}}{\left[\left\{k - m\,\omega^2\right\}^2 + (c\,\omega)^2\right]^{\frac{1}{2}}} \right\}$$

$$= \frac{(0.2)\sqrt{k^2 + c^2\,(157.08)^2}}{\left[\left\{k - 2000\,(157.08)^2\right\}^2 + c^2\,(157.08)^2\right]^{\frac{1}{2}}} = 0.1 \text{ m} \qquad (1)$$

Let $k = 5\,(10^6)$ N/m. Then Eq. (1) gives:

$$\frac{\sqrt{25\,(10^{12}) + 2.4674\,(10^4)\,c^2}}{\sqrt{1966.7717\,(10^{12}) + 2.4674\,(10^4)\,c^2}} = 0.5$$

i.e., $1.85055\,(10^4)\,c^2 = 466.6929\,(10^{12})$ i.e., $c = 158805.0$ N–s/m

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3.28

$$\frac{X}{Y} = \left[\frac{k^2 + c^2\,\omega^2}{\left(k - m\,\omega^2\right)^2 + c^2\,\omega^2}\right]^{\frac{1}{2}} \text{.sp}$$

$$\text{or} \quad \frac{10^{-6}}{Y} = \left[\frac{(10^6) + (10^3\,(200\,\pi))^2}{\left\{10^6 - \left(\frac{5000}{9.81}\right)(200\,\pi)^2\right\}^2 + \left\{(10^3)\,(200\,\pi)\right\}^2}\right]^{\frac{1}{2}}$$

or $Y = 169.5294\,(10^{-6})$ m

3.29 Equation of motion:

$$I_0\,\ddot{\theta} + \left(k\,\frac{\ell}{4}\,\theta\right)\frac{\ell}{4} + \left(c\,\frac{\ell}{4}\,\dot{\theta}\right)\frac{\ell}{4} + \left(k\,\frac{3\,\ell}{4}\,\theta\right)\frac{3\,\ell}{4} = M_0 \cos\omega t$$

$$\text{or} \quad I_0\,\ddot{\theta} + c\,\frac{\ell^2}{16}\,\dot{\theta} + \frac{5}{8}\,k\,\ell^2\,\theta = M_0 \cos\omega t$$

$$\text{where } I_0 = \frac{m\,\ell^2}{12} + m\,(\frac{\ell}{4})^2 = \frac{7}{48}\,m\,\ell^2 = \frac{7}{48}\,(10)\,(1^2) = 1.4583 \text{ kg--m}^2$$

$$\frac{c\,\ell^2}{16} = \frac{(1000)\,(1^2)}{16} = 62.5 \text{ N--m--s/rad}$$

$$\frac{5}{8}\,k\,\ell^2 = \frac{5}{8}\,(5000)\,(1^2) = 3125.0 \text{ N--m/rad}$$

$$\omega = \frac{1000\,(2\,\pi)}{60} = 104.72 \text{ rad/sec}$$

Equation of motion becomes:

$$1.4583\,\ddot{\theta} + 62.5\,\dot{\theta} + 3125.0\,\theta = 100 \cos 104.72\,t$$

Steady state response is given by Eq. (3.28):

$$\theta_p(t) = \Theta \cos(\omega\,t - \phi) = \Theta \cos(104.72\,t - \phi).sp$$

$$\text{where} \quad \Theta = \frac{100}{\left[\left\{3125.0 - 1.4583\,(104.72^2)\right\}^2 + \left\{62.5\,(104.72)\right\}^2\right]^{\frac{1}{2}}} = 0.006927 \text{ rad}$$

$$\text{and} \quad \phi = \tan^{-1}\left(\frac{62.5\,(104.72)}{3125.0 - 1.4583\,(104.72^2)}\right) = -0.4705 \text{ rad} = -26.9606^\circ$$

3.30 $m = 100$ kg, $F_0 = 100$ N, $X_{max} = 0.005$ m at $\omega = 300$ rpm $= 31.416$ rad/sec. Equations (3.33) and (3.34) yield:

$$\omega = \omega_n \sqrt{1 - 2\,\zeta^2} = \sqrt{\frac{k}{m}}\,\sqrt{1 - 2\,\zeta^2} = 31.416$$

$$\text{or} \quad k\,(1 - 2\,\zeta^2) = (100)\,(31.416^2) = 98{,}696.5056 \tag{1}$$

$$\text{and} \quad X_{max} = \delta_{st}\,\frac{1}{2\,\zeta\sqrt{1-\zeta^2}} = \frac{F_0}{k}\,\frac{1}{2\,\zeta\sqrt{1-\zeta^2}} = 0.005$$

$$\text{or} \quad k\,\zeta\sqrt{1-\zeta^2} = \frac{F_0}{2\,(0.005)} = 10{,}000.0 \tag{2}$$

Divide Eq. (1) by (2):

$$\frac{1 - 2\,\zeta^2}{\zeta\sqrt{1-\zeta^2}} = 9.8696 \tag{3}$$

Squaring Eq. (3) and rearranging leads to:

$$101.4090\,\zeta^4 - 101.4090\,\zeta^2 + 1 = 0 \quad \text{or} \quad \zeta = 0.0998,\ 0.9950$$

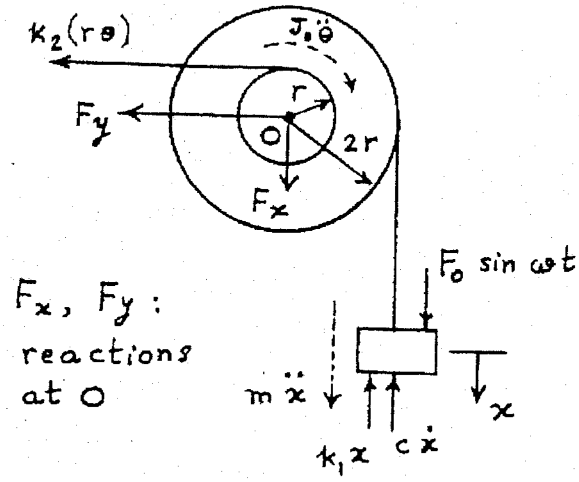
Using $\zeta = 0.0998$ in Eq. (1), we obtain

$$k = \frac{98696.5056}{1 - 2\,(0.0998^2)} = 100{,}702.4994 \text{ N/m}$$

Since $\zeta = \dfrac{c}{2\,m\,\omega_n}$, we find

$$c = 2\,m\,\omega_n\,\zeta = 2\,(100)\sqrt{\frac{100702.4944}{1000}}\,(0.0998) = 633.4038 \text{ N--s/m}$$

3.31



Equation of motion for rotation of pulley about O:

$$-k_2\,(\theta\,r)\,r - J_0\,\ddot{\theta} - k_1\,x\,(2\,r) - c\,\dot{x}\,(2\,r) + F_0 \sin \omega\,t\,(2\,r) - m\,\ddot{x}\,(2\,r) = 0 \tag{1}$$

where $\theta = x/(2r)$. Equation (1) can be rearranged as:

$$\left(\frac{J_0}{2\,r} + 2\,m\,r\right)\ddot{x} + 2\,c\,r\,\dot{x} + \left(2\,k_1\,r + \frac{1}{2}\,k_2\,r\right)x = 2\,r\,F_0 \sin \omega\,t \tag{2}$$

For given data, Eq. (2) becomes

$$11\,\ddot{x} + 50\,\dot{x} + 112.5\,x = 5 \sin 20\,t \tag{3}$$

Steady state response is given by Eq. (3.25):

$$x_p(t) = X \cos(\omega\,t - \phi)$$

where $X = \dfrac{5}{\left[\left\{112.5 - 11\,(20^2)\right\}^2 + \left\{50\,(20)\right\}^2\right]^{\frac{1}{2}}} = 0.001136$ m

and $\phi = \tan^{-1}\left(\dfrac{50\,(20)}{112.5 - 11\,(20^2)}\right) = -\,0.2291$ rad $= -\,13.1287°$

3.32 (a)

$$\sum M_0 = 0 \text{ (about hinge):}$$

$$I_0\,\ddot{\theta} + \left(k\,\theta\,\frac{3\,\ell}{4}\right)\frac{3\,\ell}{4} + (c\,\ell\,\dot{\theta})\,\ell = \frac{\ell}{2}\,F_0 \sin \omega\,t$$

$$\text{or}\quad I_0\,\ddot{\theta} + c\,\ell^2\,\dot{\theta} + \frac{9}{16}\,k\,\ell^2\,\theta = \frac{F_0\,\ell}{2} \sin \omega\,t$$

Magnitude of steady state response:

$$\Theta_a = \left(\frac{F_0\,\ell}{2}\right) \Bigg/ \left[\left\{\frac{9}{16}\,k\,\ell^2 - I_0\,\omega^2\right\}^2 + (c\,\ell^2\,\omega)^2\right]^{\frac{1}{2}} \tag{1}$$

(b)

$$\sum M_0 = 0 \text{ (about hinge):}$$

$$I_0 \ddot{\theta} + (k \ell \theta) \ell + \left(c \frac{3\ell}{4} \dot{\theta} \right) \frac{3\ell}{4} = \frac{\ell}{2} F_0 \sin \omega t$$

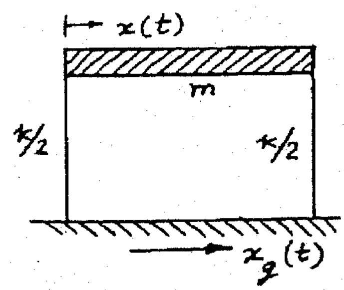
$$\text{or} \quad I_0 \ddot{\theta} + \frac{9}{16} c \ell^2 \dot{\theta} + k \ell^2 \theta = \frac{F_0 \ell}{2} \sin \omega t$$

Magnitude of steady state response:

$$\Theta_b = \left(\frac{F_0 \ell}{2} \right) \Big/ \left[\left\{ k \ell^2 - I_0 \omega^2 \right\}^2 + \left\{ \frac{9}{16} c \ell^2 \omega \right\}^2 \right]^{\frac{1}{2}} \tag{2}$$

Usually, c is small compared to k. If the term containing c is negligible, Θ_a will be smaller than Θ_b. Hence arrangement (a) is desirable.

3.33 $\ddot{y}(t) = \ddot{x}_g(t) = A \cos \omega t \quad ; \quad \dot{y}(t) = \frac{A}{\omega} \sin \omega t + B_1$



$y(t) = -\frac{A}{\omega^2} \cos \omega t + B_1 t + B_2$

Assuming $y(0) = \dot{y}(0) = 0$, we get

$y(t) = -\frac{A}{\omega^2} \cos \omega t$

Equation of motion:

$m \ddot{x} + k (x - y) = 0$

i.e., $m \ddot{z} + k z = -m \ddot{y} = -m \ddot{x}_g(t) = -m A \cos \omega t$

where $z = x - y$

Solution is : $z(t) = \dfrac{-m A \cos \omega t}{k - m \omega^2}$

$\therefore \quad x(t) = z(t) + y(t) = -\left(\dfrac{m}{k - m\omega^2} + \dfrac{1}{\omega^2} \right) A \cos \omega t$

3.34 From solution of problem 3.33,

$$x(t) = \left| \frac{-m A}{k - m \omega^2} \right| \sin \omega t - \frac{A}{\omega^2} \sin \omega t$$

$$= \left| \frac{-2000 \left(\frac{100}{1000} \right)}{0.1 * 10^6 - 2000 (25)^2} \right| \sin 25t - \left(\frac{100}{1000} \right) \frac{1}{(25)^2} \sin 25t$$

For maximum $x(t)$,

$$x(t) = \left(\frac{-200}{1.15 \times 10^6} - \frac{1}{6250} \right) \sin 25t = -3.3391 \times 10^{-4} \sin 25t \text{ m}$$

$\therefore$ Maximum horizontal displacement of floor = 0.3339 mm

3.35 $m(\ddot{x}-\ddot{y}) + k(x-y) = -m\ddot{y} = -m\ddot{x}_g$ (E₁)

Here $y(t) = x_g(t) = X_g \cos\omega t$, and Eq. (E₁) becomes

$$m\ddot{z} + kz = m\omega^2 X_g \cos\omega t \quad \text{with} \quad z = x - y$$

Solution is:

$$z(t) = \frac{m\omega^2 X_g \cos\omega t}{k - m\omega^2} = \frac{X_g r^2 \cos\omega t}{1-r^2}$$

with $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.5\times 10^6}{2000}} = 15.8114$ rad/sec

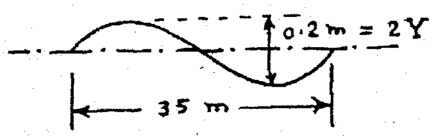
and $r = \omega/\omega_n = 200/15.8114 = 12.6491$

$$z(t) = \left(\frac{15}{1000}\right)\left\{\frac{12.6491^2}{1-12.6491^2}\right\} \cos 200t = -0.01509 \cos 200t \text{ m}$$

$$x(t) = y(t) + z(t) = \{0.015 \cos 200t - |0.01509| \cos 200t\} \text{ m}$$

∴ Amplitude of vibration of floor = 0.03009 m = 30.09 mm.

3.36 Time taken by car to travel one cycle (35 m) is



$$\tau = \frac{35 \times 3600}{60 \times 1000} = 2.1 \text{ sec}$$

Excitation frequency $= \omega = \frac{2\pi}{\tau} = 2.992$ rad/sec

$\omega_n = 2\pi(2) = 12.5664$ rad/sec, $r = \frac{\omega}{\omega_n} = 0.2381$, $\zeta = 0.15$

Amplitude of vibration of car is given by Eq. (3.68):

$$\frac{X}{Y} = \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2} \quad (E_1)$$

$$X = 0.1 \left\{\frac{1+(2\times 0.15\times 0.2381)^2}{(1-0.2381^2)^2 + (2\times 0.15\times 0.2381)^2}\right\}^{1/2}$$

$$= 0.105977 \text{ m}$$

The most unfavorable speed corresponds to the maximum of $\frac{X}{Y}$ in Eq. (E₁). For maximum of $\frac{X}{Y}$ with respect to r,

$$\frac{d}{dr}\left[\frac{1+4\zeta^2 r^2}{1+r^4-2r^2+4\zeta^2 r^2}\right] = 0$$

i.e., $$\frac{(1+r^4-2r^2+4\zeta^2 r^2)(8\zeta^2 r) - (1+4\zeta^2 r^2)(4r^3-4r+8\zeta^2 r)}{(1+r^4-2r^2+4\zeta^2 r^2)^2} = 0$$

i.e., $$-4r(2\zeta^2 r^4 + r^2 - 1) = 0$$

i.e., $$r = 0 \quad \text{or} \quad r^2 = \frac{-1 \pm \sqrt{1+8\zeta^2}}{4\zeta^2}$$

Feasible value of $r^2 = \dfrac{-1 + \sqrt{1+8(0.15)^2}}{4(0.15)^2} = 0.9586$

$r = \dfrac{\omega}{\omega_n} = 0.9791$

$\omega = 0.9791\,(12.5664) = 12.3035$ rad/sec $= \dfrac{2\pi}{\tau}$

where $\tau = \dfrac{35 \times 3600}{s \times 1000}$ and s = speed of car in km/hr.

$\therefore \quad s = \dfrac{12.3035 \times 35 \times 3.6}{2\pi} = 246.7279$ km/hr.

(3.37) Equations (3.73) and (3.68) give

$$F_T = m\omega^2 X = m\omega^2 Y \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}$$

$$\frac{F_T}{kY} = \frac{m\omega^2}{k}\left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}$$

$$= r^2\left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}$$

(3.38) Eq. (3.75): $\quad m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = m\omega^2 Y \cos\omega t$

steady-state solution is:

$$z(t) = \frac{m\omega^2 Y \cos(\omega t - \phi_1)}{(k-m\omega^2)^2 + (c\omega)^2} = Z\cos(\omega t - \phi_1)$$

where $\phi_1 = \tan^{-1}\left(\dfrac{c\omega}{k-m\omega^2}\right)$

Damping force $= c\dfrac{dz}{dt} = -c\omega Z\sin(\omega t - \phi_1)$

Energy absorbed per cycle by the damper (E):

$$E = \int_{t=0}^{2\pi/\omega} c\frac{dz}{dt}\cdot dz = \int_0^{2\pi/\omega}\{-c\omega Z\sin(\omega t-\phi_1)\}\{-\omega Z\sin(\omega t-\phi_1)\}\,dt$$

$$= c\,\omega^2 Z^2\int_0^{2\pi/\omega}\sin^2(\omega t-\phi_1)\,dt = \pi c\omega Z^2$$

Since $Z = m\omega^2 Y \Big/ \sqrt{(k-m\omega^2)^2+(c\omega)^2}$,

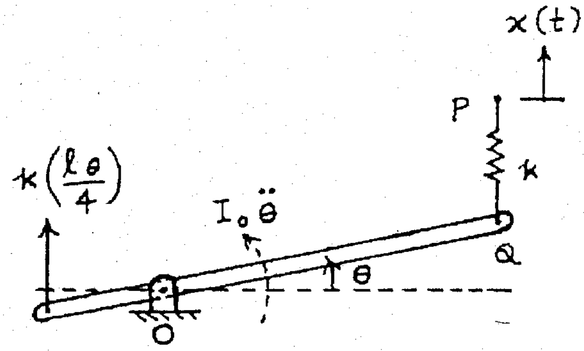
$$E = \left\{\frac{\pi c\omega\,(m^2\omega^4 Y)}{(k-m\omega^2)^2 + c^2\omega^2}\right\}$$

For maximum power, $\dfrac{dE}{dc} = 0$

i.e., $$\frac{\{(k-m\omega^2)^2+c^2\omega^2\}(\pi\omega^5 m^2 Y)-\pi c\,\omega^5 m^2 Y\,(2c\,\omega^2)}{\{(k-m\omega^2)^2+c^2\omega^2\}^2}=0$$

i.e., $$c=\left(\frac{k-m\omega^2}{\omega}\right).$$

(3.39)



Linear displacement of point Q due to $\theta = \frac{3\,\ell}{4}\,\theta$ and net compression of spring PQ $= \frac{3}{4}\,\ell\,\theta - x(t)$. Equation of motion:

$$I_0\,\ddot{\theta} = -\frac{k\,\ell\,\theta}{4}\,\frac{\ell}{4} - k\left[\frac{3\,\ell\,\theta}{4} - x(t)\right]\frac{3\,\ell}{4} \tag{1}$$

where $I_0 = \frac{1}{12}\,m\,\ell^2 + m\,(\frac{\ell}{4})^2 = \frac{7}{48}\,m\,\ell^2 = \frac{7}{48}\,(10)\,(1^2) = 1.4583$ kg–m^2

Hence Eq. (1) can be rewritten as

$$I_0\,\ddot{\theta} + \left(\frac{5}{8}\,k\,\ell^2\right)\theta = \left(\frac{3}{4}\,k\,\ell\,x_0\right)\sin\omega\,t \tag{2}$$

Steady state angular displacement of the bar is given by Eq. (3.6):

$$\Theta = \left(\frac{3}{4}\,k\,\ell\,x_0\right) \Big/ \left(\frac{5}{8}\,k\,\ell^2 - I_0\,\omega^2\right) \tag{3}$$

$$= \left(\frac{3}{4}\,(1000)\,(1)\,(0.01)\right) \Big/ \left(\frac{5}{8}\,(1000)\,(1^2) - 1.4583\,(10^2)\right) = 0.01565 \text{ rad}$$

and hence $\theta(t) = \Theta\sin\omega\,t = 0.01565\sin 10\,t$ rad

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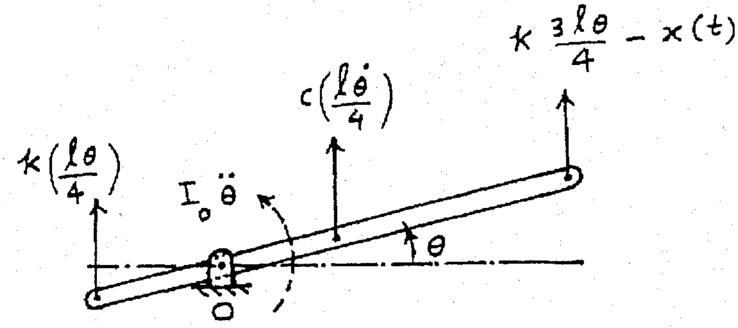
(3.40) Equation of motion:

$$I_0\,\ddot{\theta} = -k\,\frac{\ell\,\theta}{4}\,(\frac{\ell}{4}) - c\,\frac{\ell}{4}\,\dot{\theta}\,(\frac{\ell}{4}) - k\left[\frac{3\,\ell}{4}\,\theta - x(t)\right]\frac{3\,\ell}{4}$$

i.e., $$I_0\,\ddot{\theta} + \frac{1}{16}\,c\,\ell^2\,\dot{\theta} + \frac{5}{8}\,k\,\ell^2 = \frac{3}{4}\,k\,\ell\,x(t) = \frac{3}{4}\,k\,\ell\,x_0\sin\omega\,t \tag{1}$$

where $$I_0 = \frac{1}{12}\,m\,\ell^2 + m\left(\frac{\ell}{4}\right)^2 = \frac{7}{48}\,m\,\ell^2 = \frac{7}{48}\,(10)\,(1^2) = 1.4583 \text{ kg–m}^2 \tag{2}$$



Using given data, Eq. (1) can be expressed as

$$1.4583\,\ddot{\theta} + \frac{1}{16}\,(500)\,(1^2)\,\dot{\theta} + \frac{5}{8}\,(1000)\,(1^2)\,\theta = \frac{3}{4}\,(1000)\,(1)\,(0.01)\sin 10\,t$$

$$\text{i.e.,}\quad 1.4583\,\ddot{\theta} + 31.25\,\dot{\theta} + 625.0\,\theta = 7.5\sin 10\,t \tag{3}$$

Steady state angular displacement of the bar is given by Eq. (3.28) with:

$$\Theta = \frac{7.5}{\left\{\left[625.0 - 1.4583\,(10^2)\right]^2 + 31.25^2\,(10^2)\right\}^{\frac{1}{2}}} = 0.01311 \text{ rad}$$

$$\phi = \tan^{-1}\left(\frac{31.25\,(10)}{625.0 - 1.4583\,(10^2)}\right) = 0.5779 \text{ rad}$$

$$\therefore\ \theta(t) = \Theta\sin(\omega\,t - \phi) = 0.01311\sin(10\,t - 0.5779) \text{ rad}$$

3.41 Displacement transmissibility (T):

$$T = \frac{X}{Y} = \left[\frac{1 + (2\,\zeta\,r)^2}{(1 - r^2)^2 + (2\,\zeta\,r)^2}\right]^{\frac{1}{2}}$$

For maximum of T,

$$\frac{dT}{dr} = \frac{1}{2}\left[\frac{1 + 4\,\zeta^2\,r^2}{(1 + r^4 - 2\,r^2) + 4\,\zeta^2\,r^2}\right]^{-\frac{1}{2}}$$

$$\frac{\left[(1 - r^2)^2 + (2\,\zeta\,r)^2\right](8\,\zeta^2\,r) - (1 + 4\,\zeta^2\,r^2)\left[4\,r^3 - 4\,r + 8\,\zeta^2\,r\right]}{\left[(1 - r^2)^2 + (2\,\zeta\,r)^2\right]^2} = 0$$

This equation can be simplified to obtain:

$$(2\,\zeta^2)\,r^4 + r^2 - 1 = 0$$

$$\text{Solution:}\quad r^2 = \frac{-1 \pm \sqrt{1 + 8\,\zeta^2}}{4\,\zeta^2}$$

$$\text{or}\quad r = r_m = \frac{1}{2\,\zeta}\sqrt{\sqrt{1 + 8\,\zeta^2} - 1}$$

(3.42) Equation of motion: $M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin\omega t$

where $\omega = \dfrac{3000\,(2\,\pi)}{60} = 314.16$ rad/sec, $M = 100$ kg, $c = 2000$ N-s/m, $k = 10^6$ N/m, $m = 0.1$ kg and $e = r = 0.1$ m. Steady state response is:

$$x_p(t) = X \sin(\omega t - \phi)$$

.spwhere $X = \dfrac{m\,e\,\omega^2}{\left[(k - M\,\omega^2)^2 + (c\,\omega)^2\right]^{\frac{1}{2}}}$

$$= \frac{0.1\,(0.1)\,(314.16^2)}{\left[\left\{10^6 - 100\,(314.16^2)\right\}^2 + (2000\,(314.16))^2\right]^{\frac{1}{2}}} = 110.9960\,(10^{-6})\text{ m}$$

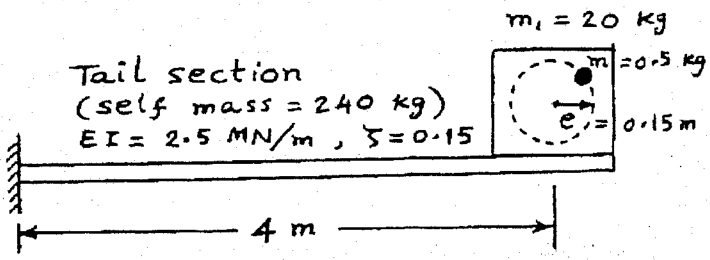
and $\phi = \tan^{-1}\left(\dfrac{c\,\omega}{k - M\,\omega^2}\right) = \tan^{-1}\left(\dfrac{2000\,(314.16)}{10^6 - 100\,(314.16^2)}\right)$

$= -\,0.07072$ rad $= -\,4.0520°$

(3.43) k = spring constant of cantilever beam

$$= \frac{3EI}{l^3} = \frac{3\,(2.5\times 10^6)}{4^3}$$

$$= 0.1172\times 10^6\ \text{N/m}$$



$$\omega_n = \sqrt{\frac{k}{m_1 + 0.25\,m_b}} = \sqrt{\frac{0.1172\times 10^6}{20 + 0.25\,(240)}} = 38.2753\ \text{rad/sec}$$

$\omega = 2\pi\,(1500)/60 = 157.08$ rad/sec

$r = \omega/\omega_n = 157.08/38.2753 = 4.1040$, $r^2 = 16.8428$

Forced response is given by Eq. (3.79) :

$$x_p(t) = X \sin(\omega t - \phi)$$

where

$$X = \frac{m\,e}{m_1}\cdot\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$= \frac{(0.5)(0.15)}{20}\;\frac{16.8428}{\sqrt{(1 - 16.8428)^2 + (2\times 0.15\times 4.1040)^2}}$$

$$= 3.9747\times 10^{-3}\ \text{m} = 3.9747\ \text{mm}$$

(3.44) $\delta_{st} = \frac{45}{1000}\,m = \frac{Mg}{k} = \frac{380 \times 9.81}{k}$

i.e., $k = 82{,}840$ N/m

$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{82{,}840}{380}} = 14.7648$ rad/sec ; $\omega = \frac{2\pi(1750)}{60}$

$= 183.26$ rad/sec

$r = \frac{\omega}{\omega_n} = \frac{183.26}{14.7648} = 12.412$; $r^2 = 154.0566$

(i) Amplitude of vibration

$$X = \frac{me}{M} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.15}{380} \frac{154.0566}{\sqrt{(153.0566)^2 + 0}}$$

$= 3.9732 \times 10^{-4}$ m

(ii) Force transmitted to ground

$= kX = (82840)(3.9732 \times 10^{-4}) = 32.9140$ N

(3.45) $I = \frac{1}{12}(0.5)(0.1)^3 = 0.4167 \times 10^{-4}$ m^4

$$k = \frac{192\,EI}{l^3} = \frac{192\,(2.07 \times 10^{11})(0.4167 \times 10^{-4})}{(5)^3} = 1.3248 \times 10^7 \text{ N/m}$$

(a) $\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{13.248 \times 10^6}{75}} = 420.2856$ rad/sec

$\omega = 2\pi(1200)/60 = 125.664$ rad/sec

$r = \omega/\omega_n = 125.664/420.2856 = 0.299$, $r^2 = 0.0894$

Amplitude of steady-state vibration is given by Eq. (3.30) with $\zeta = 0$:

$$X = \frac{\delta_{st}}{|r^2 - 1|} = \frac{F_0}{k|(r^2-1)|} = \frac{5000}{(1.3248 \times 10^7)(0.9106)}$$

$= 0.4145 \times 10^{-3}$ m

(b) Using the effective mass due to self weight of beam (for a cantilever) to be valid here also,

$$\omega_n = \sqrt{\frac{k}{M + 0.2357\,m}}$$

where M = mass of motor = 75 kg, and

m = mass of beam = $(5 \times 0.5 \times 0.1)\left(\frac{76.5 \times 10^3}{9.81}\right) = 1949.5313$ kg

$$\omega_n = \sqrt{\frac{13.248 \times 10^6}{75 + (1949.5313)(0.2357)}} = 157.4339 \text{ rad/sec}$$

$r = \omega/\omega_n = 125.664/157.4339 = 0.7982$, $r^2 = 0.6371$

$$X = \frac{\delta_{st}}{|r^2-1|} = \frac{F_0}{k\,|r^2-1|} = \frac{5000}{(1.3248\times10^7)(0.3629)}$$

$$= 1.0400 \times 10^{-3}\ m$$

3.46 Let width $= 0.5\ m$ and thickness $= t\ m$.

$$I = \frac{1}{12}(0.5)\,t^3 = \frac{t^3}{24}\quad m^4$$

$$k = \frac{3EI}{l^3} = \frac{3\,(2.07\times10^{11})\,(t^3/24)}{(5)^3} = 2.07\times10^8\ t^3\quad N/m$$

$$\omega_n = \sqrt{\frac{k}{M + 0.2357\,m}}$$

where m = mass of beam $= (5\times0.5\times t)\left(\frac{76.5\times10^3}{9.81}\right) = 19495.41\,t$ kg

$$\omega_n = \sqrt{\frac{2.07\times10^8\ t^3}{75 + 0.2357\,(19495.41\,t)}}$$

$$r = \frac{\omega}{\omega_n} = 125.664\sqrt{\frac{75 + 4595.0688\,t}{2.07\times10^8\ t^3}}$$

$$X = \frac{\delta_{st}}{|r^2-1|} = \frac{F_0}{k\,|r^2-1|}$$

i.e.,
$$0.5 = \frac{5000}{(2.07\times10^8\,t^3)\left\{(125.664)^2\left[\frac{75+4595.0688\,t}{2.07\times10^8\,t^3}\right]-1\right\}}$$

i.e., $1.3108\times10^4\,t^3 - 4595.069\,t - 74.367 = 0$

By trial and error, the value of t is found as

$$t \simeq 0.6\ m.$$

Since this is too large, we can start with a new width such as $1.0\ m$.

3.47 $m = (600/9.81)$ N, $\quad \omega = 2\pi(1000)/60 = 104.72$ rad/sec

$k = 6\,(6000) = 36{,}000$ N/m

$$\omega_n = \sqrt{k/m} = \sqrt{36000/\left(\frac{600}{9.81}\right)} = 24.2611\ \text{rad/sec}$$

$$r = \omega/\omega_n = 104.72/24.2611 = 4.3164,\qquad r^2 = 18.6311$$

$$X = \frac{F_0}{k\,|r^2-1|} = \frac{m_0\,e\,\omega^2}{k\,|r^2-1|}$$

where m_0 = unbalanced mass and e = eccentricity

i.e.,
$$2.5 \times 10^{-3} = \frac{m_o e \ (104.72)^2}{36000 \ |17.6311|}$$

i.e., $m_o e = 0.1447$ kg-m

$\therefore$ Unbalance $= W_o e = m_o g e = 0.1447 (9.81) = 1.4195$ N-m

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3.48 $m = \dfrac{1000}{386.4} = 2.588 \ \dfrac{lb - s^2}{in}$, $\omega = \dfrac{2\pi (1500)}{60} = 157.08 \ \dfrac{rad}{s}$

Possible isolators are:
(i) $k = 45\,000$ lb/in , $\zeta = 0$
(ii) $k = 90\,000$ lb/in , $\zeta = 0$
(iii) $k = 45\,000$ lb/in , $\zeta = 0.15$
(iv) $k = 90\,000$ lb/in , $\zeta = 0.15$

We will compare the force transmissibilities of these isolators.

$$\text{Force transmissibility} = T_r = \sqrt{\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}}$$

(i) $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{45000}{2.588}} = 131.8634$ rad/sec

$r = \omega/\omega_n = 157.08/131.8634 = 1.1912$, $r^2 = 1.4190$

$$T_r = \frac{1}{|1-r^2|} = \frac{1}{0.419} = 2.3866$$

(ii) $\omega_n = \sqrt{\dfrac{90\,000}{2.588}} = 186.4829$ rad/sec

$r = \omega/\omega_n = 157.08/186.4829 = 0.8423$, $r^2 = 0.7095$

$$T_r = \frac{1}{|1-r^2|} = \frac{1}{0.2905} = 3.4423$$

(iii) $\omega_n = \sqrt{\dfrac{45\,000}{2.588}} = 131.8634$ rad/sec

$r = 1.1912$, $r^2 = 1.4190$, $\zeta = 0.15$

$$T_r = \sqrt{\frac{1 + (2 \times 1.1912 \times 0.15)^2}{(1 - 1.4190)^2 + (2 \times 1.1912 \times 0.15)^2}} = 1.9282$$

(iv) $\omega_n = 186.4829$ rad/sec , $r = 0.8423$, $r^2 = 0.7095$, $\zeta = 0.15$

$$T_r = \sqrt{\frac{1 + (2 \times 0.8423 \times 0.15)^2}{(1 - 0.7095)^2 + (2 \times 0.8423 \times 0.15)^2}} = 2.6789$$

$\therefore$ Isolation (iii) is best.

3.49 Eq. (3.82): $$\frac{MX}{me} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

When $r = 1$,

$$\frac{MX}{me} = \frac{1}{2\zeta} \quad \text{or} \quad \frac{M}{me} = \frac{1}{2\zeta X} = \frac{1}{2\zeta(0.55)} = \frac{1}{1.1\zeta} \qquad (E_1)$$

When $r =$ large,

$$\frac{MX}{me} \approx 1 \quad \text{or} \quad \frac{M}{me} \approx \frac{1}{X} = \frac{1}{0.15} \qquad (E_2)$$

Combining (E_1) and (E_2), we obtain

$$\frac{M}{me} = \frac{1}{0.15} = \frac{1}{1.1\zeta}$$

$$\therefore \quad \zeta = 0.1364$$

3.50 For each spring,

$$k = \frac{G d^4}{64 n R^3} = \frac{(11.5385 \times 10^6)(0.25)^4}{64(8)(1.5)^3} = 26.083 \text{ lb/in}$$

Total $k = 4(26.083) = 104.332$ lb/in

$$\omega = \frac{2\pi(1800)}{60} = 188.496 \text{ rad/sec}$$

$m = 100/386.4$ lb-s²/in , $M = 750/386.4$ lb-s²/in , $\zeta = 0$

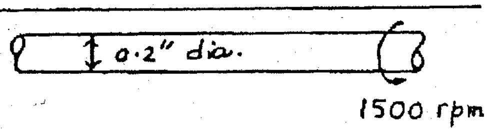
$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{104.332}{(750/386.4)}} = 7.3316 \text{ rad/sec}$$

$r = 188.496/7.3316 = 25.7102$, $r^2 = 661.0144$

$$X = \frac{me}{M} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{100(0.01)}{750}\left(\frac{661.0144}{660.0144}\right)$$

$$= 1.3354 \times 10^{-3} \text{ in.}$$

3.51 $\omega = 2\pi(1500)/60 = 157.08$ rad/sec



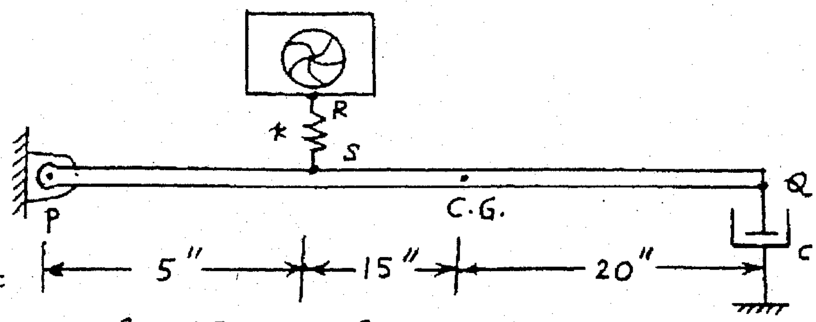
(a) Force due to eccentricity of rotor

$$= m e \omega^2 = \left(\frac{30}{386.4}\right)(0.01)(157.08)^2 = 19.1569 \text{ lb.}$$

(b) H.P. = (Force)(eccentricity)(angular velocity)

$$= (19.1569)\left(\frac{0.01}{12}\right)\left(\frac{157.08}{550}\right) = 0.004559 \text{ hp.}$$

(3.52) $\omega_{n,fan} = \sqrt{\dfrac{k}{m_{fan}}}$



$= \sqrt{\dfrac{200}{50/386.4}}$

$= 39.3141$ rad/sec

$\omega = \dfrac{2\pi(750)}{60} = 78.54$ rad/sec

$(J_p)_{plate + fan} = \frac{1}{3}\left(\frac{100}{386.4}\right)(40)^2 + \left(\frac{50}{386.4}\right)(5)^2 = 141.2612$ lb-in-sec^2

$F_0 = m e \omega^2 = \left(\frac{50}{386.4}\right)(0.1)(78.54)^2 = 79.8205$ lb

Point R is subjected to the force, $F(t) = F_0 \cos \omega t = 79.8205 \cos 78.54 t$ lb.

Assume that S is not moving.

Then R is displaced by :

$$x(t) = \frac{F_0 \cos \omega t}{|k - m\omega^2|} = \frac{F_0 \cos \omega t}{k\left|1 - \left(\frac{\omega}{\omega_n}\right)^2\right|} = \frac{79.8205 \cos \omega t}{200\left|1 - \left(\frac{78.54}{39.3141}\right)^2\right|}$$

$= 0.1334 \cos 78.54\, t$ inch

Let θ = angular displacement of plate PQ.

Displacement of $S = 5\,\theta$ inch

Extension of spring RS $= (5\theta - 0.1334 \cos 78.54 t)$ inch

Restoring moment of spring force about P

$= 200\,[5\,\theta - 0.1334 \cos 78.54\, t\,]\, 5$ lb-in

Velocity of $Q = 40\,\dot\theta$ inch/sec

Damping force at $Q = 40\,\dot\theta\,(1) = 40\,\dot\theta$ lb

Moment of damping force about $P = 40\,\dot\theta\,(40) = 1600\,\dot\theta$ lb-in

Equation of motion of plate PQ :

$J_p\,\ddot\theta + 1600\,\dot\theta + 1000\,(5\theta - 0.1334 \cos 78.54 t) = 0$

i.e., $141.2612\,\ddot\theta + 1600\,\dot\theta + 5000\,\theta = 133.4 \cos 78.54\, t$ $\quad (E_1)$

Comparing (E_1) with Eq. (3.24), the solution of (E_1) can be expressed as $\theta_p(t) = \Theta \cos(\omega t - \phi)$

where, from Eqs. (3.30) and (3.31), we get

$$\Theta = \frac{(133.4/5000)}{\sqrt{(1 - 174.2751)^2 + (2 \times 0.9519 \times 13.2013)^2}} = 1.5238 \times 10^{-4} \text{ rad}$$

and $\phi = \tan^{-1}(-25.1326/173.2751) = -8.2529°$

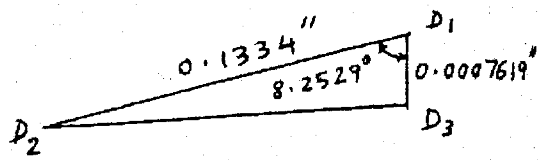
Steady state motion of $Q = \theta_p\,(40)$

$= 0.006095 \cos(78.54\, t + 8.2529°)$ inch

Displacement of S = $\Theta(5)$ inch

$= (1.5238 \times 10^{-4})(5)$ inch

$= 0.0007619$ inch



$D_2 D_3$ = maximum deformation of Spring $\approx 0.1334''$

Max. force transmitted to point $S = k(D_2 D_3)$

$= 200(0.1334) = 26.68$ lb

(3.53) $$I = \int_0^{2\pi/\omega} \sin \omega t \cdot \cos(\omega t - \phi)\, dt$$

$$= \int_0^{2\pi/\omega} \sin \omega t \left[\cos \omega t \cdot \cos \phi + \sin \omega t \cdot \sin \phi\right] dt$$

$$= \int_0^{2\pi/\omega} \left\{\cos \phi\, (\sin \omega t \cdot \cos \omega t) + \sin \phi\, (\sin^2 \omega t)\right\} dt$$

$$= \int_0^{2\pi/\omega} \left\{\cos \phi \left(\frac{\sin 2\omega t}{2}\right) + \sin \phi \left(\frac{1 - \cos 2\omega t}{2}\right)\right\} dt$$

$$= \frac{\cos \phi}{2}\left(-\frac{\cos 2\omega t}{2\omega}\right)\Bigg|_0^{2\pi/\omega} + \frac{\sin \phi}{2}\left(t - \frac{\sin 2\omega t}{2}\right)\Bigg|_0^{2\pi/\omega}$$

$$= \frac{\pi}{\omega} \sin \phi$$

$$\Delta W' = \omega F_0 X \cdot I = \omega F_0 X \sin \phi$$

(3.54) Let $x(t)$ = displacement of mass m

New length of each spring, k_1 $= (l^2 + x^2)^{1/2}$

New tension in each spring k_1 $= T = (\sqrt{l^2 + x^2} - l)\, k_1 + T_0$

Horizontal component of new tension in each spring k_1

$= T x / \sqrt{l^2 + x^2}$

Vertical component of new tension in each spring $k_1 = \dfrac{T l}{\sqrt{l^2 + x^2}}$

Total friction force $= \mu\, mg + \dfrac{2 T l}{\sqrt{l^2 + x^2}}$

<u>when mass moves to right:</u>

Equation of motion of mass m:

$$m\ddot{x} + k_2 x + \frac{2 T x}{\sqrt{l^2 + x^2}} - \mu\left[mg + \frac{2 T l}{\sqrt{l^2 + x^2}}\right] = p_0 A \sin \omega t$$

where A = area of piston.

i.e., $$m\ddot{x} + x\left(k_2 + 2\frac{T_0}{l}\right) = \mu m g + 2\mu T_0 + p_0 A \sin \omega t$$

<u>Similarly, when the mass moves to left:</u>

$$m\ddot{x} + x\left(k_2 + 2\frac{T_0}{l}\right) = -\mu m g - 2\mu T_0 + p_0 A \sin \omega t$$

(3.55) $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2100}{2}} = 32.403703$ rad/sec

N = vertical force = $mg = 2\,(9.81) = 19.62$ N

$$\frac{\omega}{\omega_n} = \frac{2.5173268 \times 2\pi}{32.403703} = 0.4881191$$

$$X = \frac{F_0}{k}\left[\frac{1 - \left(\frac{4\mu N}{\pi F_0}\right)^2}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2}\right]^{1/2}$$

i.e., $$0.075 = \frac{120}{2100}\left[\frac{1 - \left\{\frac{4\mu\,(19.62)}{\pi\,(120)}\right\}^2}{(1 - 0.4881191^2)}\right]^{1/2}$$

i.e., $$1.3125 = \left(\frac{1 - 0.04334\ \mu^2}{0.5802473}\right)^{1/2}$$

i.e., $$0.9995666 = 1 - 0.04334\ \mu^2$$

i.e., $$\mu = 0.1$$

(3.56) (a) $k = \frac{W}{\delta_{st}} = \frac{5000}{0.05} = 10^5$ N/m

When $\omega = \omega_n$, Eq. (3.102) gives

$$X = \frac{F_0}{k\beta} \Rightarrow 0.1 = \frac{1000}{(10^5)\,\beta} \Rightarrow \beta = 0.1$$

(b) $\Delta W = \pi\, c_{eq}\, \omega\, X^2 = \pi \beta k X^2$ where $c_{eq} = \frac{\beta k}{\omega}$ from Eq. (3.100)

$\Delta W = \pi\,(0.1)\,(10^5)\,(0.1)^2 = 314.16$ Joules/cycle

(c) Steady state amplitude at one-quarter of resonant frequency:

$$\frac{\omega}{\omega_n} = 0.25$$

$$X = \frac{F_0}{k\left[\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \beta^2\right]^{1/2}} = \frac{1000}{10^5\left[\left\{1 - 0.25^2\right\}^2 + (0.1)^2\right]^{1/2}}$$

$= 0.01061$ m

(d) Steady state amplitude at thrice the resonant frequency:

$$\frac{\omega}{\omega_n} = 3$$

$$X = \frac{1000}{10^5\left[(1 - 3^2)^2 + (0.1)^2\right]^{1/2}} = 0.00125 \text{ m}$$

(3.57) $\Delta W = \pi \beta * X^{\gamma}$

$$\left.\begin{array}{l} 3.8 = \pi \beta (60\,000)(0.04)^{\gamma} \\ 9.5 = \pi \beta (60000)(0.06)^{\gamma} \end{array}\right\} \Rightarrow \begin{array}{l} \beta (0.04)^{\gamma} = 0.0000202 \\ \beta (0.06)^{\gamma} = 0.0000504 \end{array}$$

Taking logarithms,

$$\ln \beta + \gamma \ln (0.04) = \ln (0.0000202)$$
$$\ln \beta + \gamma \ln (0.06) = \ln (0.0000504)$$

i.e., $\ln \beta - 3.218876\ \gamma = -10.809828 \quad ---- (E_1)$

$\ln \beta - 2.813411\ \gamma = -9.895511 \quad ---- (E_2)$

subtracting (E_1) from (E_2), $\quad 0.405465\ \gamma = 0.914309$

$\gamma = 2.254964$

From (E_1), $\ln \beta = -10.809828 + 3.218876\ (2.254964) = -3.551378$

$\beta = 0.028685$

(3.58) Work done $= W = \int F\, dx = \int F\, \dot{x}\, dt$

If $F(t) = F_0 \cos \omega t$ and $x(t) = X \cos(\omega t - \phi)$, work done in one cycle

$$= W = -\int_0^{2\pi/\omega} F_0 \cos \omega t \,.\, \omega X \sin (\omega t - \phi)\, dt$$

$$= -\frac{F_0 \omega X \cos \phi}{2} \left(-\frac{1}{2\omega} \cos 2\omega t \right)_0^{2\pi/\omega} + \frac{F_0 \omega X \sin \phi}{2} \left(t + \frac{1}{2\omega} \sin 2\omega t \right)_0^{\frac{2\pi}{\omega}}$$

$$= F_0 \pi X \sin \phi$$

Given data: $F_0 = 5$ lb, $\omega = 3\pi \frac{\text{rad}}{\text{sec}}$, $\tau = \frac{2}{3}$ sec, $\phi = \frac{\pi}{3}$, $X = 0.5''$

$$W = F_0 \pi X \sin \phi = 5\pi (0.5) \sin \frac{\pi}{3} = 6.8018 \text{ lb-in}$$

(i) In one second, it will complete $1\frac{1}{2}$ cycles.

$$W\Big|_{1 \text{ second}} = 1.5\ W = 10.2027 \text{ lb-in.}$$

(ii) In four seconds, it will complete 6 cycles.

$$W\Big|_{4 \text{ seconds}} = 6\ W = 40.8108 \text{ lb-in.}$$

(3.59) Damping force $= F = c (\dot{x})^n$

Energy dissipated per quarter cycle during harmonic motion $x(t) = X \sin \omega t$ is

$$\frac{\Delta W}{4} = \int_0^{\pi/2\omega} c (\dot{x})^n\, dx = \int_0^{\pi/2\omega} c\, (\omega X \cos \omega t)^n\, dx$$

But $dx = \dot{x}\, dt = \omega X \cos \omega t \cdot dt$

$$\Delta W = 4c\,\omega^{n+1}\,X^{n+1}\int_0^{\pi/2\omega}\cos^{n+1}\omega t\cdot dt$$

$$= 4c\,\omega^{n+1}\,X^{n+1}\left\{\frac{1}{(n+1)\omega}\cos^n\omega t\cdot\sin\omega t\,\Big|_0^{\pi/2\omega}+\frac{n}{n+1}\int_0^{\pi/2\omega}\cos^{n-1}\omega t\cdot dt\right\}$$

$$= 4c\,\omega^{n+1}\,X^{n+1}\left(\frac{n}{n+1}\right)\int_0^{\pi/2\omega}\cos^{n-1}\omega t\cdot dt$$

Equating this expression to $\pi\, c_{eq}\,\omega\, X^2$, we obtain

$$c_{eq} = \frac{4c\,\omega^n\,X^{n-1}}{\pi}\left(\frac{n}{n+1}\right)\int_0^{\pi/2\omega}\cos^{n-1}\omega t\cdot dt \equiv c\,\omega^n\,X^{n-1}\,\alpha_n$$

where $\alpha_n = \dfrac{4}{\pi}\left(\dfrac{n}{n+1}\right)\displaystyle\int_0^{\pi/2\omega}\cos^{n-1}\omega t\cdot dt$ ---- (E_1)

For example, for $n=2$, (E_1) becomes

$$\alpha_n = \frac{4}{\pi}\left(\frac{2}{3}\right)\int_0^{\pi/2\omega}\cos\omega t\cdot dt = \frac{8}{3\pi}\left(\frac{\sin\omega t}{\omega}\right)_0^{\pi/2\omega} = \frac{8}{3\pi\omega}$$

and hence $c_{eq} = \dfrac{8c\omega X}{3\pi}$

which can be seen to be same as the expression found in Example 3.6.

For few other values of n, α_n can be found as follows:

n	1	2	3	4
α_n	$\frac{1}{\omega}$	$\frac{8}{3\pi\omega}$	$\frac{3}{4\omega}$	$\frac{32}{15\pi\omega}$

The amplitude can be found as

$$X = \frac{F_0}{\sqrt{(k-m\omega^2)^2+c_{eq}^2\,\omega^2}} = \frac{F_0}{\sqrt{k^2(1-r^2)^2+c_{eq}^2\,\omega^2}}$$

$$= \frac{F_0}{\sqrt{k^2(1-r^2)^2+c^2\,\omega^{2(n+1)}\,X^{2(n-1)}\,\alpha_n^2}}$$

(3.60) Energy dissipated per cycle for viscous damping $= \pi c\,\omega X^2$

Energy dissipated per cycle for Coulomb damping $= 4\mu N X$

Equivalent viscous damping constant (c_{eq}) is given by

$$\pi\, c_{eq}\,\omega X^2 = \pi c\,\omega X^2 + 4\mu N X$$

$$c_{eq} = \left(c + \frac{4\mu N}{\pi\omega X}\right)$$

Amplitude X is given by

$$X = \frac{F_0}{\sqrt{(k - m\,\omega^2)^2 + c_{eq}^2\,\omega^2}} = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c_{eq}^2\,\omega^2}}$$

substituting for c_{eq}, squaring and rearranging,

$$X^2\left\{k^2(1-r^2)^2 + c^2\omega^2\right\} + X\left(\frac{8\mu N c\,\omega}{\pi}\right) + \left(\frac{16\,\mu^2 N^2}{\pi^2} - F_0^2\right) = 0$$

(3.61) (a) Equation of motion $\quad m\ddot{x} \pm \mu N + c(\dot{x})^3 + k\,x = F_0 \cos\omega t$

Thus the system has combined Coulomb and velocity-cubed damping.

For Coulomb damping, $\quad c_{eq\,1} = \dfrac{4\mu N}{\pi\,\omega\,X}$ $\quad (E_1)$

For velocity-cubed damping, the equivalent viscous damping coefficient can be obtained from the solution of problem 3.59:

$$c_{eq\,2} = c\,\omega^3 X^2 \alpha_3 \qquad (E_2)$$

where

$$\alpha_3 = \frac{4}{\pi}\left(\frac{3}{4}\right)\int_0^{\pi/2\omega} \cos^2\omega t\; dt = \frac{3}{4\,\omega} \qquad (E_3)$$

$$\therefore \quad c_{eq\,2} = \frac{3}{4}\,c\,\omega^2 X^2 \qquad (E_4)$$

and

$$c_{eq} = c_{eq\,1} + c_{eq\,2} = \frac{4\mu N}{\pi\,\omega\,X} + \frac{3}{4}\,c\,\omega^2 X^2 \qquad (E_5)$$

(b) Steady state amplitude under harmonic force:

$$X = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c_{eq}^2\,\omega^2}} = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + \left\{\frac{4\mu N}{\pi\omega X} + \frac{3}{4}c\omega^2 X^2\right\}^2\omega^2}} \qquad (E_6)$$

(c) Amplitude ratio:

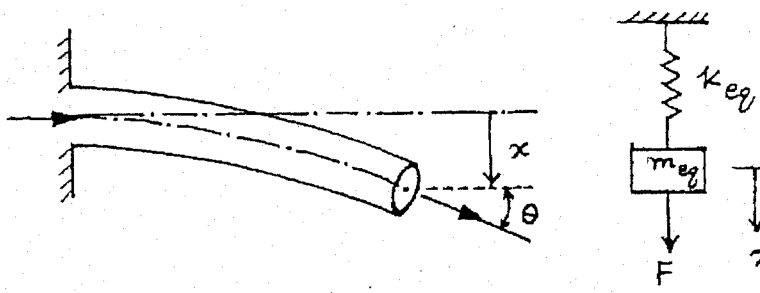
$$\frac{X}{\delta_{st}} = \frac{X}{(F_0/k)} = \frac{1}{\sqrt{(1-r^2)^2 + \left(\frac{c_{eq}^2\,\omega^2}{k^2}\right)}}$$

$$= \frac{1}{\sqrt{(1-r^2)^2 + \left\{\frac{4\mu N}{\pi X k} + \frac{3}{4}\,\frac{c\,\omega^3 X^2}{k}\right\}^2}} \qquad (E_7)$$

At resonance, $r = 1$ and Eq. (E_7) reduces to

$$\left.\frac{X}{\delta_{st}}\right|_{resonance} = \frac{1}{\left\{\frac{4\mu N}{\pi X k} + \frac{3}{4k}\,c\,\omega^3 X^2\right\}} \qquad (E_8)$$

(3.62) Model the pipe as a single degree of freedom system with m_{eq} = equivalent mass at end $= \dfrac{33}{140}$ m (m = mass of pipe; see Problem 2.38) and $k_{eq} = \dfrac{3\,E\,I}{\ell^3}$. Slope of pipe at end:



$$\theta = \frac{F \ell^2}{2 E I} = \frac{F \ell^3}{3 E I} \left(\frac{3}{2 \ell} \right) = \frac{3 x}{2 \ell}$$

where x = end deflection of the cantilever pipe under a transverse load F. Force induced due to fluid velocity v is $\rho A v^2$. Force acting on the single degree of freedom system (in vertical direction):

$$F = \rho A v^2 \sin \theta \approx \rho A v^2 \theta = \rho A v^2 \frac{3 x}{2 \ell}$$

$$\text{Equation of motion:} \quad m_{eq} \ddot{x} + k_{eq} x = F$$

$$\text{or} \quad \frac{33}{140} m \ddot{x} + \left(\frac{3 E I}{\ell^3} - \frac{3 \rho A v^2}{2 \ell} \right) x = 0$$

$$\text{Instability occurs when} \quad \frac{3 E I}{\ell^3} - \frac{3 \rho A v^2}{2 \ell} < 0 \quad \text{or} \quad v > \sqrt{\frac{2 E I}{\rho A \ell^2}}$$

3.63 Assume Reynolds number (R_e) greater than 1000. Strouhal number (St) for vortex shedding is taken as: $St = \frac{f d}{V} = 0.21$ where f = frequency of vortex shedding, d = diameter of cylinder and V = velocity of fluid (air). At 50 mph speed,

$$V = \frac{50 (1760) (36)}{3600} = 880 \text{ in/sec} \quad \text{and} \quad f = \frac{0.21 V}{d} = \frac{184.8}{d} \text{ Hz} \ \text{(d in inches)}$$

For the three sections of the antenna, the vortex frequencies are:

$$f_1 = \frac{184.8}{0.3} = 616.0 \text{ Hz}$$
$$f_2 = \frac{184.8}{0.2} = 924.0 \text{ Hz}$$
$$f_3 = \frac{184.8}{0.1} = 1848.0 \text{ Hz}$$

At 75 mph speed,

$$V = \frac{75 (1760) (36)}{3600} = 1320 \text{ in/sec} \quad \text{and} \quad f = \frac{0.21 V}{d} = \frac{277.2}{d} \text{ Hz} \ \text{(d in inches)}$$

For the three sections of the antenna, the frequencies are:

$$f_1 = \frac{277.2}{0.3} = 924.0 \text{ Hz}$$
$$f_2 = \frac{277.2}{0.2} = 1386.0 \text{ Hz}$$

$$f_3 = \frac{277.2}{0.1} = 2772.0 \text{ Hz}$$

Since the natural frequencies are much smaller, no instability occurs.

3.64 (a) Equivalent mass of single d.o.f. system $= m_{eq} = M + \frac{33}{140}$ m where m = mass of cylindrical part of the sign post:

$$m = \frac{\pi}{4}(D^2 - d^2)\, h\, \rho = \frac{\pi}{4}(0.25^2 - 0.2^2)(10)\left(\frac{76500}{9.81}\right) = 1378.0527 \text{ kg}$$

$$\therefore m_{eq} = 200 + \frac{33}{140}(1378.0527) = 524.8267 \text{ kg}$$

$$I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{4}(0.25^4 - 0.2^4) = 113.208\,(10^{-6}) \text{ m}^4$$

Equivalent stiffness of the system:

$$k_{eq} = \frac{3\,E\,I}{h^3} = \frac{3\,(207\,(10^9))\,(113.208\,(10^{-6}))}{10^3} = 70{,}302.168 \text{ N/m}$$

Natural frequency of transverse vibration of sign post:

$$\omega_1 = \left(\frac{k_{eq}}{m_{eq}}\right)^{\frac{1}{2}} = \left(\frac{70302.168}{524.8267}\right)^{\frac{1}{2}} = 11.5738 \text{ rad/sec} = 1.8420 \text{ Hz}$$

(b) Wind velocity corresponding to maximum vibration of sign post (V) is given by:

$$St = 0.21 = \frac{f_1\,D}{V} \quad \text{or} \quad V = \frac{f_1\,D}{0.21} = \frac{(1.8420)\,(0.25)}{0.21} = 2.1929 \text{ m/s}$$

(c) Maximum force acting on the system due to wind velocity:

$$F(t) = F_0 \sin \omega t = \frac{1}{2}\, c\, \rho\, V^2 A \sin \omega t = \frac{1}{2}(1)(1.2215)(2.19299^2)(8.0) \sin \omega t \text{ N}$$
$$= 23.4958 \sin \omega t \text{ N}$$

where c = 1 for a cylinder, ρ = density of air = 1.2215 kg/m^3, A = projected area of cylindrical part = (0.8)(10) = 8.0 m^2, and ω = frequency of wind force. Equation of motion:

$$m_{eq}\,\ddot{x} + c_{eq}\,\dot{x} + k_{eq}\,x = F(t)$$

and the maximum steady state displacement of the sign post occurs when $\omega = \omega_1$ and is given by Eq. (3.34):

$$X = \frac{\delta_{st}}{2\,\varsigma} = \frac{F_0}{k_{eq}\,(2)\,\varsigma} = \frac{23.4958}{2\,(0.1)\,(70302.168)} = 0.001671 \text{ m}$$

(3.65) (a) Equation of motion $\quad m\ddot{x} + c\dot{x} + k x = F_0 x$

or $\quad m\ddot{x} + c\dot{x} + (k - F_0)\, x = 0 \qquad (E_1)$

Assuming the solution $\quad x(t) = C\, e^{st} \qquad (E_2)$

where C is a constant, Eq. (E_1) gives the auxiliary equation

$$s^2 + \frac{c}{m} s + \left(\frac{k - F_0}{m}\right) = 0 \qquad (E_3)$$

Roots of (E_3) are

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k - F_0}{m}\right)} \qquad (E_4)$$

First consider the case of positive stiffness $(k > F_0)$. For this case, following possibilities exist.

1. If $\left(\frac{c}{2m}\right)^2 > \left(\frac{k - F_0}{m}\right)$:

   Both s_1 and s_2 will be real and negative and hence

   $$x(t) = C_1\, e^{s_1 t} + C_2\, e^{s_2 t} \qquad (E_5)$$

   will be stable.

2. If $\left(\frac{c}{2m}\right)^2 = \left(\frac{k - F_0}{m}\right)$:

   Both s_1 and s_2 will be identical, real and negative.

   Solution $\quad x(t) = (C_1 + C_2\, t)\, e^{s_1 t} \qquad (E_6)$

   will be stable since $e^{s_1 t} \to 0$ as $t \to \infty$.

3. If $\left(\frac{c}{2m}\right)^2 < \left(\frac{k - F_0}{m}\right)$:

   Here s_1 and s_2 will be complex conjugates and solution will be

   $$x(t) = C\, e^{-\left(\frac{c}{2m}\right)t} \sin\left(\sqrt{\left\{-\left(\frac{c}{2m}\right)^2 + \left(\frac{k - F_0}{m}\right)\right\}}\, t + \phi\right) \qquad (E_7)$$

   This represents a converging oscillatory motion and hence the system will be stable.

Next consider the case of negative stiffness $(k < F_0)$. Here

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 + \left(\frac{F_0 - k}{m}\right)} \qquad (E_8)$$

Thus s_1 will be positive and s_2 will be negative, and the solution becomes

$$x(t) = C_1\, e^{+s_1 t} + C_2\, e^{-|s_2| t} \qquad (E_9)$$

This solution can be seen to diverge as $t \to \infty$.

(b) Equation of motion $\quad m\ddot{x} + c\dot{x} + k x = F_0 \dot{x}$

or $\quad \ddot{x} + \left(\frac{c - F_0}{m}\right)\dot{x} + \frac{k}{m} x = 0 \qquad (E_{10})$

Assuming $x(t) = C\,e^{st}$ the auxiliary equation becomes

$$s^2 + \left(\frac{c - F_0}{m}\right) s + \frac{k}{m} = 0 \qquad (E_{11})$$

and hence

$$s_{1,2} = -\left(\frac{c - F_0}{2m}\right) \pm \sqrt{\left(\frac{c - F_0}{2m}\right)^2 - \frac{k}{m}} \qquad (E_{12})$$

First consider the case of positive damping ($c > F_0$) in (E_{10}). For this case, it can be seen that the system will be stable for all possible values of $\left\{ \left(\frac{c - F_0}{2m}\right)^2 - \frac{k}{m} \right\}$.

Next, consider the case of negative damping ($c < F_0$). Depending on the sign of the quantity under the radical in Eq. (E_{12}), we will have three types of solution.

1. $\left(\frac{c - F_0}{2m}\right)^2 > \frac{k}{m}$. Here both s_1 and s_2 are real and positive and hence $x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$ $\qquad (E_{13})$

   This denotes a diverging nonoscillatory motion ; so the system is unstable.

2. $\left(\frac{c - F_0}{2m}\right)^2 = \frac{k}{m}$. Here s_1 and s_2 are identical and are real and positive. Hence $x(t) = (C_1 + C_2\,t)\,e^{s_1 t}$ $\qquad (E_{14})$

   This represents a diverging nonoscillatory solution; so the system will be unstable.

3. $\left(\frac{c - F_0}{2m}\right)^2 < \frac{k}{m}$. Here s_1 and s_2 are complex conjugates and hence

   $$s_{1,2} = \left(\frac{F_0 - c}{2m}\right) \pm i \sqrt{\frac{k}{m} - \left(\frac{c - F_0}{2m}\right)^2} \qquad (E_{15})$$

   The solution becomes

   $$x(t) = X\, e^{\left(\frac{F_0 - c}{2m}\right) t} \sin\left(\sqrt{\frac{k}{m} - \left(\frac{c - F_0}{2m}\right)^2}\; t + \phi\right) \qquad (E_{16})$$

   Since the exponent is positive, Eq. (E_{16}) denotes a diverging oscillatory motion and hence the system is unstable.

Thus the condition for dynamic stability of the system can be stated as

$$F_0 \leq c \qquad (E_{17})$$

9	0.15013780E-01	-0.29585633E+00	-0.15013779E+01
10	0.51364703E-02	-0.32777125E+00	-0.51364702E+00
11	-0.52436423E-02	-0.32760152E+00	0.52436423E+00
12	-0.15110461E-01	-0.29536372E+00	0.15110462E+01
13	-0.23498155E-01	-0.23421355E+00	0.23498156E+01
14	-0.29585691E-01	-0.15013662E+00	0.29585693E+01
15	-0.32777146E-01	-0.51363401E-01	0.32777145E+01
16	-0.32760132E-01	0.52437652E-01	0.32760131E+01
17	-0.29536312E-01	0.15110572E+00	0.29536314E+01
18	-0.23421254E-01	0.23498255E+00	0.23421254E+01
19	-0.15013559E-01	0.29585743E+00	0.15013559E+01
20	-0.51362254E-02	0.32777163E+00	0.51362258E+00

3.67 Eq. (3-35) gives the complete solution

$$x(t) = X_0\, e^{-\zeta\omega_n t} \cos(\omega_d t + \phi_0) + X \cos(\omega t - \phi) \qquad (E.1)$$

Differentiation gives

$$\dot{x}(t) = -\zeta\omega_n X_0\, e^{-\zeta\omega_n t} \cos(\omega_d t + \phi_0) - \omega_d X_0\, e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_0) - \omega X \sin(\omega t - \phi) \qquad (E.2)$$

$$\ddot{x}(t) = X_0\, e^{-\zeta\omega_n t} (\zeta^2\omega_n^2 - \omega_d^2) \cos(\omega_d t + \phi_0) + 2\zeta\omega_n\omega_d X_0\, e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_0) - \omega^2 X \cos(\omega t - \phi) \qquad (E.3)$$

Let $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ be known. Then

$$x_0 = X_0 \cos\phi_0 + X \cos\phi \qquad (E.4)$$

$$\dot{x}_0 = -\zeta\omega_n X_0 \cos\phi_0 - \omega_d X_0 \sin\phi_0 + \omega X \sin\phi \qquad (E.5)$$

Eqs. (E.4) and (E.5) give

$$X_0 = \left[(X_0 \cos\phi_0)^2 + (X_0 \sin\phi_0)^2\right]^{1/2} = \left[(x_0 - X\cos\phi)^2 + \left\{\frac{-\dot{x}_0 - \zeta\omega_n x_0 + \zeta\omega_n X \cos\phi + \omega X \sin\phi}{\omega_d}\right\}^2\right]^{1/2} \qquad (E.6)$$

$$\phi_0 = \tan^{-1}\left\{\frac{-\dot{x}_0 - \zeta\omega_n x_0 + \zeta\omega_n X\cos\phi + \omega X \sin\phi}{\omega_d (x_0 - X\cos\phi)}\right\} \qquad (E.7)$$

Now the computer program TOTALR can be written. In addition to the arguments used in section 3.12, the following arguments are also used:

XO = value of $x(t)$ at $t = 0$. Input

XDO = value of $\frac{dx}{dt}(t)$ at $t = 0$. Input.

The program and the output are given below.

9	0.15013780E-01	-0.29585633E+00	-0.15013779E+01
10	0.51364703E-02	-0.32777125E+00	-0.51364702E+00
11	-0.52436423E-02	-0.32760152E+00	0.52436423E+00
12	-0.15110461E-01	-0.29536372E+00	0.15110462E+01
13	-0.23498155E-01	-0.23421355E+00	0.23498156E+01
14	-0.29585691E-01	-0.15013062E+00	0.29585693E+01
15	-0.32777146E-01	-0.51363401E-01	0.32777145E+01
16	-0.32760132E-01	0.52437652E-01	0.32760131E+01
17	-0.29536312E-01	0.15110572E+00	0.29536314E+01
18	-0.23421254E-01	0.23498255E+00	0.23421254E+01
19	-0.15013559E-01	0.29585743E+00	0.15013559E+01
20	-0.51362254E-02	0.32777163E+00	0.51362258E+00

3.67 Eq. (3-35) gives the complete solution

$$x(t) = X_0 \, e^{-\zeta\omega_n t} \cos(\omega_d t + \phi_0) + X \cos(\omega t - \phi) \qquad \text{(E.1)}$$

Differentiation gives

$$\dot{x}(t) = -\zeta\omega_n X_0 \, e^{-\zeta\omega_n t} \cos(\omega_d t + \phi_0) - \omega_d X_0 \, e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_0) - \omega X \sin(\omega t - \phi) \qquad \text{(E.2)}$$

$$\ddot{x}(t) = X_0 \, e^{-\zeta\omega_n t} (\zeta^2\omega_n^2 - \omega_d^2) \cos(\omega_d t + \phi_0) + 2\zeta\omega_n\omega_d X_0 \, e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_0) - \omega^2 X \cos(\omega t - \phi) \qquad \text{(E.3)}$$

Let $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ be known. Then

$$x_0 = X_0 \cos\phi_0 + X \cos\phi \qquad \text{(E.4)}$$

$$\dot{x}_0 = -\zeta\omega_n X_0 \cos\phi_0 - \omega_d X_0 \sin\phi_0 + \omega X \sin\phi \qquad \text{(E.5)}$$

Eqs. (E.4) and (E.5) give

$$X_0 = \left[(X_0 \cos\phi_0)^2 + (X_0 \sin\phi_0)^2 \right]^{1/2}$$

$$= \left[(x_0 - X\cos\phi)^2 + \left\{ \frac{-\dot{x}_0 - \zeta\omega_n x_0 + \zeta\omega_n X \cos\phi + \omega X \sin\phi}{\omega_d} \right\}^2 \right]^{1/2} \qquad \text{(E.6)}$$

$$\phi_0 = \tan^{-1} \left\{ \frac{-\dot{x}_0 - \zeta\omega_n x_0 + \zeta\omega_n X \cos\phi + \omega X \sin\phi}{\omega_d (x_0 - X\cos\phi)} \right\} \qquad \text{(E.7)}$$

Now the computer program TOTALR can be written. In addition to the arguments used in section 3.12, the following arguments are also used:

XO = value of $x(t)$ at $t=0$. Input

XDO = value of $\frac{dx}{dt}(t)$ at $t=0$. Input.

The program and the output are given below.

```
      EX1=EXP(-XAI*OMN*TIME)
      EC1=COS(OMD*TIME+PZ)
      EC2=COS(OM*TIME-XPHI)
      ES1=SIN(OMD*TIME+PZ)
      ES2=SIN(OM*TIME-XPHI)
      X(I)=XZ*EX1*EC1+XAMP*EC2
      XD(I)=-XAI*OMN*XZ*EX1*EC1-OMD*XZ*EX1*ES1-OM*XAMP*ES2
10    XDD(I)=XZ*EX1*EC1*((XAI*OMN)**2-OMD**2)+2.0*XAI*OMN*OMD*XZ*
     2 EX1*ES1-(OM**2)*XAMP*EC2
      RETURN
      END
```

```
TOTAL RESPONSE OF AN UNDERDAMPED
SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE


GIVEN DATA:
XM = 0.10000000E+02
XC = 0.45000000E+02
XK = 0.25000000E+04
F0 = 0.18000000E+03
OM = 0.21991199E+02
N  =20

INITIAL CONDITIONS:
X0  = 0.15000000E-01
XD0 = 0.50000000E+01


RESPONSE:

  I      X(I)             XD(I)              XDD(I)

  1  -0.10720424E-01  -0.62618327E+01   0.14583885E+02
  2  -0.13126461E+00  -0.56405468E+01   0.46769913E+02
  3  -0.23313771E+00  -0.44189043E+01   0.73777733E+02
  4  -0.30557248E+00  -0.27362981E+01   0.92193657E+02
  5  -0.34123138E+00  -0.79179698E+00   0.99568497E+02
  6  -0.33717862E+00   0.11797444E+01   0.94842743E+02
  7  -0.29529962E+00   0.29382858E+01   0.78578743E+02
  8  -0.22207454E+00   0.42714534E+01   0.52945923E+02
  9  -0.12771350E+00   0.50244703E+01   0.21447788E+02
 10  -0.24761723E-01   0.51222720E+01  -0.11575108E+02
 11   0.73627755E-01   0.45800018E+01  -0.41590317E+02
 12   0.15549700E+00   0.34999285E+01  -0.64561966E+02
 13   0.21168864E+00   0.20550721E+01  -0.77567314E+02
 14   0.23700854E+00   0.46208435E+00  -0.79236084E+02
 15   0.23077171E+00  -0.10522565E+01  -0.69936806E+02
 16   0.19665717E+00  -0.22854314E+01  -0.51678036E+02
 17   0.14191106E+00  -0.30888264E+01  -0.27741650E+02
 18   0.76043457E-01  -0.33878446E+01  -0.21128597E+01
 19   0.92454441E-02  -0.31893401E+01   0.21192802E+02
 20  -0.49206819E-01  -0.25754893E+01   0.38788284E+02
```

```
      EX1=EXP(-XAI*OMN*TIME)
      EC1=COS(OMD*TIME+PZ)
      EC2=COS(OM*TIME-XPHI)
      ES1=SIN(OMD*TIME+PZ)
      ES2=SIN(OM*TIME-XPHI)
      X(I)=XZ*EX1*EC1+XAMP*EC2
      XD(I)=-XAI*OMN*XZ*EX1*EC1-OMD*XZ*EX1*ES1-OM*XAMP*ES2
10    XDD(I)=XZ*EX1*EC1*((XAI*OMN)**2-OMD**2)+2.0*XAI*OMN*OMD*XZ*
     2 EX1*ES1-(OM**2)*XAMP*EC2
      RETURN
      END
```

```
TOTAL RESPONSE OF AN UNDERDAMPED
SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE


GIVEN DATA:
XM = 0.10000000E+02
XC = 0.45000000E+02
XK = 0.25000000E+04
F0 = 0.18000000E+03
OM = 0.21991199E+02
N  =20

INITIAL CONDITIONS:
X0  = 0.15000000E-01
XD0 = 0.50000000E+01


RESPONSE:

  I     X(I)             XD(I)             XDD(I)

  1  -0.10720424E-01  -0.62618327E+01   0.14583885E+02
  2  -0.13126461E+00  -0.56405468E+01   0.46769913E+02
  3  -0.23313771E+00  -0.44189043E+01   0.73777733E+02
  4  -0.30557248E+00  -0.27362981E+01   0.92193657E+02
  5  -0.34123138E+00  -0.79179698E+00   0.99568497E+02
  6  -0.33717862E+00   0.11797444E+01   0.94842743E+02
  7  -0.29529962E+00   0.29382858E+01   0.78578743E+02
  8  -0.22207454E+00   0.42714534E+01   0.52945923E+02
  9  -0.12771350E+00   0.50244703E+01   0.21447788E+02
 10  -0.24761723E-01   0.51222720E+01  -0.11575108E+02
 11   0.73627755E-01   0.45800018E+01  -0.41590317E+02
 12   0.15549700E+00   0.34999285E+01  -0.64561966E+02
 13   0.21168864E+00   0.20550721E+01  -0.77567314E+02
 14   0.23700854E+00   0.46208435E+00  -0.79236084E+02
 15   0.23077171E+00  -0.10522565E+01  -0.69936806E+02
 16   0.19665717E+00  -0.22854314E+01  -0.51678036E+02
 17   0.14191106E+00  -0.30888264E+01  -0.27741650E+02
 18   0.76043457E-01  -0.33878446E+01  -0.21128597E+01
 19   0.92454441E-02  -0.31893401E+01   0.21192802E+02
 20  -0.49206819E-01  -0.25754893E+01   0.38788284E+02
```

9	-0.10027631E+00	0.46169037E+00	0.40110523E+02
10	-0.88234901E-01	0.10588365E+01	0.35293961E+02
11	-0.67556389E-01	0.15523360E+01	0.27022554E+02
12	-0.40264975E-01	0.18938812E+01	0.16105991E+02
13	-0.90321312E-02	0.20500395E+01	0.36128526E+01
14	0.23084890E-01	0.20055246E+01	-0.92339563E+01
15	0.52942146E-01	0.17646941E+01	-0.21176857E+02
16	0.77617034E-01	0.13511224E+01	-0.31046814E+02
17	0.94694205E-01	0.80529296E+00	-0.37877682E+02
18	0.10250200E+00	0.18063448E+00	-0.41000801E+02
19	0.10027614E+00	-0.46170485E+00	-0.40110458E+02
20	0.88234514E-01	-0.10588492E+01	-0.35293804E+02

(3.69) Equation of motion is $m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$

when $y(t) = Y \sin \omega t$, $x_p(t) = X \cos(\omega t - \phi_1 - \phi_2)$

Complete solution can be expressed as

$$x(t) = X_0 \, e^{-\zeta\omega_n t} \cos(\omega_d t + \phi_0) + X \cos(\omega t - \phi_1 - \phi_2) \qquad (E.1)$$

with $X = Y \left[\dfrac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2}$,

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right), \quad \phi_2 = \tan^{-1}\left(\frac{1}{2\zeta r}\right), \quad r = \frac{\omega}{\omega_n}.$$

If the initial conditions are known $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$,

$$x_0 = X_0 \cos\phi_0 + X\cos(\phi_1 + \phi_2)$$

and

$$\dot{x}_0 = -\zeta\omega_n X_0 \cos\phi_0 - \omega_d X_0 \sin\phi_0 - \omega X \sin(-\phi_1 - \phi_2)$$

Hence

$$X_0 = \left[\left\{x_0 - X\cos(\phi_1+\phi_2)\right\}^2 + \left\{ \frac{-\dot{x}_0 - \zeta\omega_n x_0 + \zeta\omega_n X \cos(\phi_1+\phi_2) + \omega X \sin(\phi_1+\phi_2)}{\omega_d} \right\}^2 \right]^{1/2}$$

$$\phi_0 = \tan^{-1}\left[\frac{-\dot{x}_0 - \zeta\omega_n x_0 + \zeta\omega_n X\cos(\phi_1+\phi_2) + \omega X \sin(\phi_1+\phi_2)}{\omega_d \left\{x_0 - X\cos(\phi_1+\phi_2)\right\}} \right]$$

If necessary, the velocity $\dot{x}(t)$ and acceleration $\ddot{x}(t)$ can be found from Eq. (E.1). The computer program and output are given below.

```
C ==================================================================
C
C SOLUTION OF PROBLEM 3.69
C MAIN PROGRAM WHICH CALLS BASEX
C RESPONSE OF A SINGLE D.O.F. SYSTEM SUBJECTED TO BASE EXCITATION,
C Y(T)=Y*SIN(OM*T)
C
C ==================================================================
```

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(20),XD(20),XDD(20)
      DATA XM,XC,XK,Y,OM,N/2.0,10.0,100.0,0.1,25.0,20/
      DATA X0,XD0/0.01,5.0/
C END OF PROBLEM-DEPENDENT DATA
      CALL BASEX (XM,XC,XK,Y,OM,N,X,XD,XDD,X0,XD0)
      PRINT 100
 100  FORMAT (//,33H TOTAL RESPONSE OF AN UNDERDAMPED,/,
     2 52H SINGLE D.O.F. SYSTEM UNDER HARMONIC BASE EXCITATION)
      PRINT 200, XM,XC,XK,Y,OM,N
 200  FORMAT (//,12H GIVEN DATA:,/,5H XM =,E15.8,/,5H XC =,E15.8,/,
     2 5H XK =,E15.8,/,5H Y  =,E15.8,/,5H OM =,E15.8,/,5H N  =,I2)
      PRINT 300,X0,XD0
 300  FORMAT (/,20H INITIAL CONDITIONS:,/,6H X0  =,E15.8,/,6H XD0 =,
     2 E15.8)
      PRINT 400
 400  FORMAT (//,10H RESPONSE:,//,5H   I ,3X,5H X(I),12X,6H XD(I),
     2 11X,7H XDD(I),/)
      DO 500 I=1,N
 500  PRINT 600, I,X(I),XD(I),XDD(I)
 600  FORMAT (I4,2X,E15.8,2X,E15.8,2X,E15.8)
      STOP
      END
C ======================================================================
C
C SUBROUTINE BASEX
C
C ======================================================================
      SUBROUTINE BASEX (XM,XC,XK,Y,OM,N,X,XD,XDD,X0,XD0)
      DIMENSION X(N),XD(N),XDD(N)
      OMN=SQRT(XK/XM)
      XAI=XC/(2.0*XM*OMN)
      OMD=OMN*SQRT(1.0-XAI**2)
      R=OM/OMN
      DELT=2.0*3.1416/(OMD*REAL(N))
      XAMP=Y*SQRT(1.0+(2.0*XAI*R)**2/((1.0-R**2)**2+(2.0*XAI*R)**2))
      PHI1=ATAN(2.0*XAI*R/(1.0-R*R))
      PHI2=ATAN(1.0/(2.0*XAI*R))
      XCC=XC
      TIME=0.0
      DO 10 I=1,N
      TIME=TIME+DELT
      XC=X0-XAMP*COS(PHI1+PHI2)
      XS=(-XD0-XAI*OMN*XC+OM*XAMP*SIN(PHI1+PHI2))/OMD
      XZ=SQRT(XC**2+XS**2)
      PZ=ATAN(XS/XC)
      EX=EXP(-XAI*OMN*TIME)
      CS=COS(OMD*TIME+PZ)
      SI=SIN(OMD*TIME+PZ)
      CS12=COS(OM*TIME-PHI1-PHI2)
      SI12=SIN(OM*TIME-PHI1-PHI2)
      X(I)=XZ*EX*CS+XAMP*CS12
      XD(I)=-XAI*OMN*XZ*EX*CS-OMD*XZ*EX*SI-OM*XAMP*SI12
 10   XDD(I)=XZ*EX*CS*((XAI*OMN)**2-OMD**2)+2.0*XAI*OMN*OMD*XZ*EX*SI
     2 -(OM**2)*XAMP*CS12
```

```
      XC=XCC
      RETURN
      END
```

```
TOTAL RESPONSE OF AN UNDERDAMPED
SINGLE D.O.F. SYSTEM UNDER HARMONIC BASE EXCITATION


GIVEN DATA:
XM = 0.20000000E+01
XC = 0.10000000E+02
XK = 0.10000000E+03
Y  = 0.10000000E+00
OM = 0.25000000E+02
N  =20

 INITIAL CONDITIONS:
 XO   = 0.99999998E-02
 XDO  = 0.50000000E+01


 RESPONSE:
```

I	X(I)	XD(I)	XDD(I)
1	-0.50330855E-01	-0.57580528E+01	-0.10305269E+02
2	-0.30772516E+00	-0.44899216E+01	0.62558521E+02
3	-0.43412885E+00	-0.65008521E+00	0.85064461E+02
4	-0.38465756E+00	0.22720275E+01	0.28117821E+02
5	-0.27407524E+00	0.18295681E+01	-0.40405914E+02
6	-0.24066399E+00	-0.41657627E+00	-0.39765869E+02
7	-0.28432682E+00	-0.90879965E+00	0.23411983E+02
8	-0.27970907E+00	0.14345939E+01	0.64174423E+02
9	-0.14624044E+00	0.38781688E+01	0.26183165E+02
10	0.39102390E-01	0.33305802E+01	-0.47345394E+02
11	0.12840362E+00	0.26650119E+00	-0.67513680E+02
12	0.80883794E-01	-0.18088942E+01	-0.10970811E+02
13	0.96553117E-02	-0.68976790E+00	0.50778744E+02
14	0.38462169E-01	0.18209940E+01	0.40712753E+02
15	0.14735566E+00	0.22107751E+01	-0.27525837E+02
16	0.19996722E+00	-0.31552225E+00	-0.66972618E+02
17	0.11764607E+00	-0.28227291E+01	-0.26468327E+02
18	-0.18114097E-01	-0.23140993E+01	0.45068512E+02
19	-0.63696094E-01	0.51316518E+00	0.59701672E+02
20	0.10478826E-01	0.21295214E+01	0.43720150E+00

3.70 Unbalanced force in vertical direction $= m e \omega^2 \sin \omega t$ (E_1)

Unbalanced force in horizontal direction $= 0$

Let M = total mass of the shaker

Equation of motion is $\quad M \ddot{x} + c \dot{x} + k x = m e \omega^2 \sin \omega t \quad (E_2)$

Steady state solution of (E_2) is

$$x(t) = X \sin(\omega t - \phi) \qquad (E_3)$$

where

$$X = \frac{m\ e\ r^2}{M\left[(1-r^2)^2 + (2\zeta r)^2\right]^{1/2}} \qquad (E_4)$$

and

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \qquad (E_5)$$

where

$$r = \frac{\omega}{\omega_n} = \omega\sqrt{(M/k)} \qquad (E_6)$$

Frequency range: 20 Hz to 30 Hz

i.e., 125.664 rad/sec to 188.496 rad/sec

$$\therefore \quad 125.664 \text{ rad/sec} \le \omega \le 188.496 \text{ rad/sec} \qquad (E_7)$$

$0.1'' \le X \le 0.2''$ where X is given by (E_4).

Mean power output over a time period τ is given by

$$P = \frac{1}{\tau}\int_0^\tau F(\tau)\ \frac{dx}{dt}(\tau)\ d\tau \qquad (E_9)$$

where $\tau = \frac{2\pi}{\omega}$,

$F(\tau) = m e \omega^2 \sin\omega t$, and

$\frac{dx}{dt} = \omega X \cos(\omega t - \phi)$

$$P \ge 1 \text{ hp} \qquad (E_{10})$$

$$\frac{M}{m} \ge 50 \qquad (E_{11})$$

Procedure:

Find ω, e, M, m, k and c so as to satisfy the requirements stated in (E_7), (E_8), (E_{10}) and (E_{11}).

3.71 $m = \frac{10^5}{386.4} = 258.7992$ lb-s²/in , $l = 600''$, $E = 30 \times 10^6$ psi

$$k = \frac{3EI}{l^3} = \frac{3\,(30\times10^6)}{(600)^3}\cdot\frac{\pi}{64}(D^4 - d^4) = 0.020453\,(D^4 - d^4)\ \frac{lb}{in}$$

$\omega = 2\pi(15) = 94.248$ rad/sec ; $\zeta = 0.15$

Ground acceleration $\ddot{y}(t) = 193.2 \sin 94.248\, t$ in/s² (E_1)

Equation of motion:

$$m\ddot{z} + c\dot{z} + k z = -m\ddot{y} = -50\,000 \sin 94.248\, t \qquad (E_2)$$

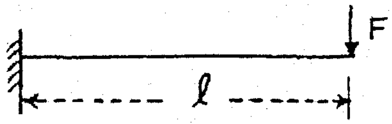
where $z = x - y$ = relative displacement.

Let $$z(t) = Z \sin(\omega t - \phi) = Z \sin(94.248\, t - \phi) \qquad (E_3)$$

with $$Z = \frac{-50\,000}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \qquad (E_4)$$

and $$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \qquad (E_5)$$

$$y_{max} = \frac{F\ell^3}{3EI} \Rightarrow \frac{3\, y_{max}}{\ell^2} = \frac{F\ell}{EI}$$



Max. bending moment $= M = F\ell$

Max. bending stress $= \dfrac{M\,(D/2)}{EI} = \dfrac{F\ell}{EI} \cdot \dfrac{D}{2} = \dfrac{3D}{2\ell^2}\, y_{max}$

If maximum relative displacement, $y_{max} = Z$, is known, max. bending stress (σ_1) is given by

$$\sigma_1 = \frac{3D}{2\ell^2} \cdot Z$$

Direct compressive stress (σ_2) due to weight of water tank is

$$\sigma_2 = \frac{mg}{\frac{\pi}{4}(D^2 - d^2)} = \frac{4 \times 10^5}{\pi\,(D^2 - d^2)}$$

Total stress $= \sigma_1 + \sigma_2 \le 30\,000$ psi

i.e., $$\frac{3D}{2\ell^2} Z + \frac{4 \times 10^5}{\pi(D^2 - d^2)} \le 30\,000 \qquad (E_6)$$

Natural frequency of water tank is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.020453\ (D^4 - d^4)}{258.7992}} \ge 30\pi \ \frac{\text{rad}}{\text{sec}}$$

i.e., $$D^4 - d^4 \ge 1.1240 \times 10^8 \qquad (E_7)$$

Weight of steel column is

$$W_s = \frac{\pi}{4}(D^2 - d^2)\,\ell\, \rho_{weight} = \frac{\pi}{4}(D^2 - d^2)(600)(0.283)$$

$$= 133.3609\ (D^2 - d^2) \text{ lb.} \qquad (E_8)$$

Problem: Find $\begin{Bmatrix} D \\ d \end{Bmatrix}$ to minimize W_s subject to restrictions of (E_6) and (E_7). Also use the conditions: $D \geq d$, $D \geq 0$ and $d \geq 0$.

Procedure: Plot the inequalities (E_6) and (E_7) in the $D-d$ space and draw contours of W_s and identify the minimum weight design.

Chapter 4

Vibration Under General Forcing Conditions

4.1 $F(t) = \frac{F_0}{\pi} + \frac{F_0}{2} \sin \omega t - \frac{2 F_0}{\pi} \sum_{n=2,4,6,\ldots} \frac{1}{(n^2-1)} \cos n\omega t$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_0}{\pi k} + \frac{F_0}{2k} \cdot \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega t - \phi_1)$$

$$- \frac{2 F_0}{\pi k} \sum_{n=2,4,6,\ldots} \frac{1}{(n^2-1)} \cdot \frac{1}{\sqrt{(1-n^2 r^2)^2 + (2\zeta n r)^2}} \cos(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$, $\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2 r^2}\right)$

4.2 From the solution of problem 1.63,

$$F(t) = \begin{cases} (2 F_0 t/\tau) & ; \quad 0 \le t \le \tau/2 \\ -(2 F_0 t/\tau) + 2 F_0 & ; \quad \tau/2 \le t \le \tau \end{cases}$$

Fourier series representation of $F(t)$ is

$$F(t) = \frac{F_0}{2} - \frac{4 F_0}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \cos n\omega t$$

$$\therefore \quad x(t) = \frac{F_0}{2k} - \frac{4 F_0}{\pi^2 k} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^2} \frac{\cos(n\omega t - \phi_n)}{\sqrt{(1-r^2 n^2)^2 + (2\zeta n r)^2}}$$

where $r = \frac{\omega}{\omega_n}$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2 r^2}\right)$.

4.3 $F(t) = \frac{8 F_0}{\pi^2} \sum_{n=1,3,5,\ldots} (-1)^{\frac{n-1}{2}} \sin \frac{n\omega t}{n^2}$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{8 F_0}{\pi^2 k} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} (-1)^{\frac{n-1}{2}} \frac{1}{\sqrt{(1-n^2 r^2)^2 + (2\zeta n r)^2}} \sin(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2 r^2}\right)$

4.4 $F(t) = \frac{F_0}{2} + \frac{F_0}{\pi} \sum_{n=1,2,\ldots}^{\infty} \frac{1}{n} \sin n\omega t$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_0}{2k} + \frac{F_0}{\pi k} \sum_{n=1,2,\ldots}^{\infty} \frac{1}{n} \cdot \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\zeta n r)^2}} \sin(n\omega t - \phi_n)$$

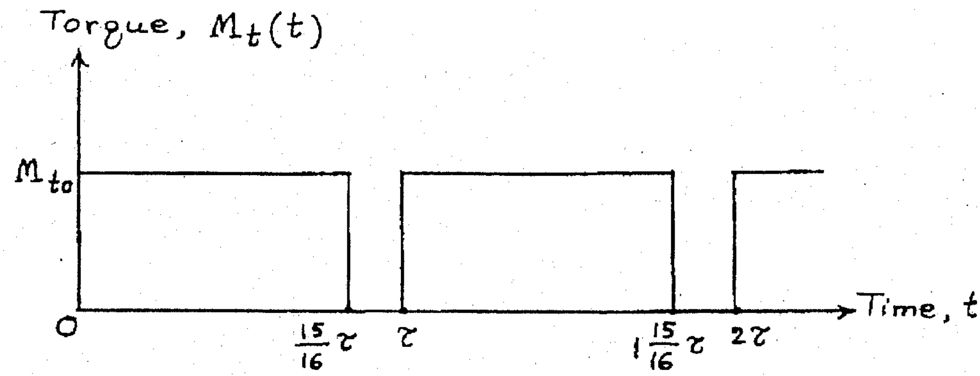
where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2r^2}\right)$

4.5 From Example 1.12, $F(t) = \frac{F_0}{\pi}\left[\frac{\pi}{2} - \sum_{n=1,2,3,\ldots} \frac{1}{n} \cdot \sin n\omega t\right]$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_0}{2k} - \frac{F_0}{\pi k} \sum_{n=1,2,3,\ldots} \frac{1}{n} \cdot \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\zeta n r)^2}} \cdot \sin(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2r^2}\right)$

4.6



Torque transmitted to driven gear is shown in the figure. It can be expressed as:

$$M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\,\omega\,t + b_n \sin n\,\omega\,t\right)$$

$$\text{where} \quad \omega = 2\,\pi\left(\frac{1000}{60}\right) = 104.72 \text{ rad/sec}$$

$$\tau = \frac{2\,\pi}{\omega} = 0.06 \text{ sec} \quad ; \quad \frac{15}{16}\,\tau = 0.05625 \text{ sec}$$

$$M_t(t) = \begin{cases} M_{t0} = 1000 \text{ N--m} \;;\; 0 \le t \le \frac{15}{16}\,\tau \\ 0 \;;\; \frac{15}{16}\,\tau \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} M_t(t)\,dt = \frac{2}{0.06}\int_0^{0.05625}(1000)\,dt = \frac{2000}{0.06}(0.05625) = 1875.0 \text{ N--m}$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} M_t(t)\cos n\,\omega\,t\,dt = \frac{2}{\tau}M_{t0}\int_0^{0.05625}\cos n\,\omega\,t\,dt$$

$$= \frac{2\,M_{t0}}{\tau}\left(\frac{\sin n\,\omega\,t}{n\,\omega}\right)_0^{0.05625} = \frac{318.3091}{n}\sin 5.8905\,n \text{ N--m}$$

$$b_n = \frac{2}{\tau} \int_0^{\tau} M_t(t) \sin n \omega t \, dt = \frac{2 M_{t0}}{\tau} \int_0^{0.05625} \sin n \omega t \, dt$$

$$= \frac{2 M_{t0}}{\tau} \left(- \frac{\cos n \omega t}{n \omega} \right)_0^{0.05625} = \frac{318.3091}{n} \left(1 - \cos 5.8905 \, n \right) \text{ N–m}$$

$$k_t = \frac{GJ}{\ell} = G \left(\frac{\pi d^4}{4} \right) \frac{1}{\ell} = (80 \, (10^9)) \left(\frac{\pi (0.05)^4}{4} \right) \frac{1}{1} = 392{,}700 \text{ N–m/rad}$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{392700}{0.1}} = 1981.6660 \text{ rad/sec}$$

Equation of motion:

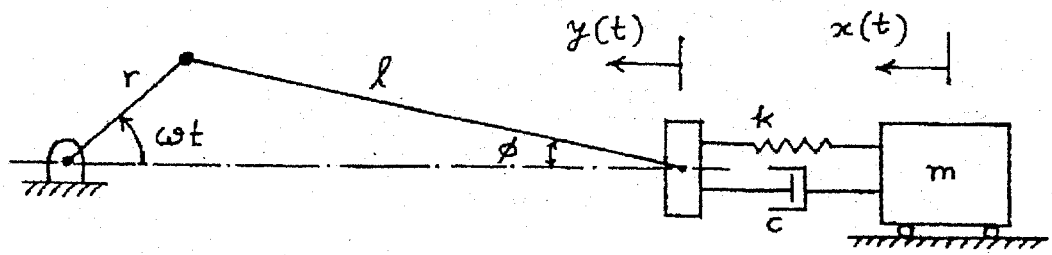
$$J_0 \ddot{\theta} + k_t \theta = M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n \omega t + b_n \sin n \omega t \right)$$

Response:

$$\theta(t) = \frac{a_0}{2 k_t} + \sum_{n=1}^{\infty} \left\{ \frac{a_n \cos n \omega t + b_n \sin n \omega t}{k_t - J_0 (n \omega)^2} \right\}$$

$$= 0.0023873 + \sum_{n=1}^{\infty} \left\{ \frac{318.3091 \sin 5.8905 \, n \cos \omega t + 318.3091 \, (1 - \cos 5.8905 \, n) \sin n \omega t}{n \, (392700.0 - 1096.6278 \, n^2)} \right\} \text{ rad}$$

4.7



Base motion is given by:

$$y(t) = r + \ell - r \cos \omega t - \ell \cos \phi = r + \ell - r \cos \omega t - \ell \sqrt{1 - \sin^2 \phi} \tag{1}$$

Using $\ell \sin \phi = r \sin \omega t$, Eq. (1) becomes

$$y(t) = r + \ell - r \cos \omega t - \ell \sqrt{1 - \frac{r^2}{\ell^2} \sin^2 \omega t} \tag{2}$$

Using the approximation:

$$\sqrt{1 - \frac{r^2}{\ell^2} \sin^2 \omega t} \approx 1 - \frac{r^2}{2 \ell^2} \sin^2 \omega t \tag{3}$$

Eq. (2) can be expressed as

$$y(t) = r + \ell - r \cos \omega t - \ell \left(1 - \frac{1}{2} \frac{r^2}{\ell^2} \sin^2 \omega t \right)$$

$$= r - r \cos \omega t + \frac{\ell}{4} \left(\frac{r}{\ell} \right)^2 - \frac{\ell}{4} \left(\frac{r}{\ell} \right)^2 \cos 2 \omega t \tag{4}$$

Equation of motion:

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$$

$$= kr - kr\cos\omega t + \frac{k\ell}{4}\left(\frac{r}{\ell}\right)^2 - \frac{k\ell}{4}\left(\frac{r}{\ell}\right)^2 \cos 2\omega t + \cdots$$

$$+ cr\omega\sin\omega t + \frac{c\ell}{4}\left(\frac{r}{\ell}\right)^2 (2\omega)\sin 2\omega t + \cdots \tag{5}$$

Solution of Eq. (5) can be found by adding the solutions due to each term on the right hand side of Eq. (5).

Solution due to constant term, F_0 (terms 1 and 3 on the r.h.s. of Eq. (5)):

$$x(t) = \frac{F_0}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\cos(\omega_d t - \phi)\right] \tag{6}$$

$$\text{where} \quad \phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

Solution due to sinusoidal term, $F_0 \sin\Omega t$ (terms 5 and 6 on the r.h.s. of Eq. (5)):

$$x(t) = X\sin(\Omega t - \phi_0) \tag{7}$$

$$\text{where} \quad X = \frac{F_0}{\left[(k - m\Omega^2)^2 + c^2\Omega^2\right]^{\frac{1}{2}}} \quad \text{and} \quad \phi_0 = \tan^{-1}\left(\frac{c\Omega}{k - m\Omega^2}\right) \tag{8}$$

Solution due to cosine term, $F_0 \cos\Omega t$ (terms 2 and 4 in Eq. (5)):

$$x(t) = X\cos(\Omega t - \phi_0) \tag{9}$$

$$\text{where} \quad X = \frac{F_0}{\left[(k - m\Omega^2)^2 + c^2\Omega^2\right]^{\frac{1}{2}}} \quad \text{and} \quad \phi_0 = \tan^{-1}\left(\frac{c\Omega}{k - m\Omega^2}\right) \tag{10}$$

For given data, $\zeta = \dfrac{c}{2\sqrt{mk}} = \dfrac{10}{2\sqrt{1\,(100)}} = 0.5$, $\dfrac{r}{\ell} = 0.1$, $\omega = 100$, $2\omega = 200$, etc. and the solution of Eq. (5) can be obtained by using Eqs. (6) to (8) suitably.

4.8 Base motion can be represented by Fourier series as (from Example 1.12):

$$y(t) = \frac{Y}{\pi}\left[\frac{\pi}{2} - \left\{\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \cdots\right\}\right] \tag{1}$$

Equation of motion of mass:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0 \tag{2}$$

Since y(t) is composed of several terms, the solution of Eq. (2) can be found by superposingg the solutions corresponding to each of the terms appearing in Eq. (1). When $y(t) = Y/2$, constant, equation of motion becomes:

$$m\ddot{x} + c\dot{x} + kx = \frac{kY}{2} = \text{constant} \tag{3}$$

The steady state solution of Eq. (3) is given by (see Example 4.3):

$$x(t) = \frac{Y}{2}\left[1 - \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \operatorname{coss}(\omega_d t - \phi)\right] \tag{4}$$

$$\text{where} \quad \phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \tag{5}$$

When $y(t) = A \sin \Omega t$, the steady state solution of Eq. (2) is given by Eq. (3.67):

$$x(t) = A \sin(\Omega t - \phi) \tag{6}$$

$$\text{where} \quad A = \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{\frac{1}{2}} \tag{7}$$

$$\phi = \tan^{-1}\left\{\frac{2\zeta r^3}{1 + r^2(4\zeta^2 - 1)}\right\} \tag{8}$$

$$\text{and} \quad r = \frac{\Omega}{\omega_n}$$

4.9 From solution of Problem 4.7, we can express the base motion as:

$$y(t) = r - r\cos\omega t + \frac{\ell}{4}\left(\frac{r}{\ell}\right)^2 - \frac{\ell}{4}\left(\frac{r}{\ell}\right)^2 \cos 2\omega t + \cdots \tag{1}$$

Equation of motion:

$$m\ddot{x} + k(x - y) \pm \mu N = 0$$

or

$$m\ddot{x} + kx \pm \mu N = ky$$

$$= \left\{kr + \frac{k\ell}{4}\left(\frac{r}{\ell}\right)^2\right\} - kr\cos\omega t - \frac{k\ell}{4}\left(\frac{r}{\ell}\right)^2 \cos 2\omega t + \cdots \tag{2}$$

For given numerical data, Eq. (2) becomes:

$$\ddot{x} + 100\,x \pm 0.981 = 100\,y$$

$$= \left\{100\,(0.1) + \frac{100\,(1)}{4}\left(\frac{0.1}{1}\right)^2\right\} - (100)\,(0.1)\cos 100\,t - \frac{100\,(1)}{4}\left(\frac{0.1}{1}\right)^2 \cos 200\,t - \cdots$$

$$= 10.25 - 10\cos 100\,t - 0.25\cos 200\,t - \cdots \tag{3}$$

Using the definition of equivalent damping constant, c_{eq}, the solution of Eq. (3) can be found by superposition.

$$c_{eq} = \frac{4\,\mu\,N}{\pi\,\Omega\,X} = \frac{4\,\mu\,m\,g}{\pi\,\Omega\,X} \tag{4}$$

where Ω is the frequency of the harmonic force and X is the amplitude of the mass.

Steady state solution due to constant term, F_0, on the r.h.s. of Eq. (3) (from Example 4.3):

$$x(t) = \frac{F_0}{k} = \left\{k\,r + \frac{k\,\ell}{4}\left(\frac{r}{\ell}\right)^2\right\} = r + \frac{\ell}{4}\left(\frac{r}{\ell}\right)^2 \tag{5}$$

Steady state solution due to harmonic term, $F_0 \cos \Omega\,t$, on the r.h.s. of Eq. (3) (from Eq. (3.93)):

$$x(t) = X\cos(\Omega\,t - \phi) \tag{6}$$

$$\text{where}\quad X = \frac{F_0}{k}\left[\frac{1 - \left(\frac{4\,\mu\,N}{\pi\,F_0}\right)^2}{\left(1 - \frac{\Omega^2}{\omega_n^2}\right)^2}\right]^{\frac{1}{2}} \quad\text{and}\quad \phi = \tan^{-1}\left\{\frac{\pm\frac{4\,\mu\,N}{\pi\,F_0}}{\left[1 - \left(\frac{4\,\mu\,N}{\pi\,F_0}\right)^2\right]^{\frac{1}{2}}}\right\} \tag{7}$$

$N = m\,g = 1\,(9.81) = 9.81$ Newtons, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{1}} = 10$ rad/sec, and + sign is to be used in Eq. (7) for $\Omega < \omega_n$ and - sign for $\Omega > \omega_n$. Equations (5) and (6) can be superposed suitably to find the complete steady state solution of Eq. (3).

4.10 Base motion can be represented by Fourier series as (see solution of Problem 4.8 or Example 1.12):

$$y(t) = \frac{Y}{\pi}\left\{\frac{\pi}{2} - \left(\sin\omega\,t + \frac{1}{2}\sin 2\,\omega\,t + \frac{1}{3}\sin 3\,\omega\,t + \cdots\right)\right\} \tag{1}$$

Equation of motion of mass:

$$m\,\ddot{x} + k\,((x - y) \pm \mu\,N = 0$$

or

$$m\ddot{x} + k x \pm \mu N = k y$$

$$= \frac{k Y}{\pi}\left[\frac{\pi}{2} - \left\{\sin \omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \cdots\right\}\right] \tag{2}$$

Using the definition of equivalent viscous damping constant:

$$c_{eq} = \frac{4\mu N}{\pi \Omega X} = \frac{4\mu m g}{\pi \Omega X} \tag{3}$$

where Ω is the frequency of the harmonic force and X is the amplitude of the mass, the solution of Eq. (2) can be determined using the superposition principle.

Steady state solution due to constant term, F_0, on the r.h.s. of Eq. (2) (from Example 4.3):

$$x(t) = \frac{F_0}{k} = \frac{k Y}{2 k} = \frac{Y}{2} \tag{4}$$

Steady state solution due to harmonic term, $F_0 \sin \Omega t$, on the r.h.s. of Eq. (2) (from Eq. (3.93)):

$$x(t) = X \sin(\Omega t - \phi)$$

$$\text{where} \quad X = \frac{F_0}{k}\left[\frac{1 - \left(\frac{4\mu N}{\pi F_0}\right)^2}{\left(1 - \frac{\Omega^2}{\omega_n^2}\right)^2}\right]^{\frac{1}{2}} \quad \text{and} \quad \phi = \tan^{-1}\left\{\frac{\pm\frac{4\mu N}{\pi F_0}}{\left[1 - \left(\frac{4\mu N}{\pi F_0}\right)^2\right]^{\frac{1}{2}}}\right\} \tag{6}$$

and + sign is to be used in Eq. (6) for $\Omega < \omega_n$ and - sign for $\Omega > \omega_n$. Equations (4) and (5) can be superposed suitably to find the complete steady state solution of Eq. (2).

4.11 From solution of problem 1.70,

$$F(t) = 9.9584 - 20.1587 \cos 10.472 t + 23.5253 \sin 10.472 t$$
$$+ 3.3099 \cos 20.944 t + 12.2646 \sin 20.944 t$$
$$+ 3.7719 \cos 31.416 t - 0.4064 \sin 31.416 t \tag{E_1}$$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{15000}{1}} = 122.4745$ rad/sec ; $\zeta = 0.1$; $r = 0.0855$

The steady state solution is given by Eq. (4.13):

$$x_p(t) = \frac{9.9584}{k} - \frac{20.1587}{k}\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\cos(10.472t-\phi_1)$$

$$+ \frac{23.5253}{k}\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(10.472t-\phi_1)$$

$$+ \frac{3.3099}{k}\frac{1}{\sqrt{(1-4r^2)^2+(4\zeta r)^2}}\cos(20.944t-\phi_2)$$

$$+ \frac{12.2646}{k}\frac{1}{\sqrt{(1-4r^2)^2+(4\zeta r)^2}}\sin(20.944t-\phi_2)$$

$$+ \frac{3.7719}{k}\frac{1}{\sqrt{(1-9r^2)^2+(6\zeta r)^2}}\cos(31.416t-\phi_3)$$

$$- \frac{0.4064}{k}\frac{1}{\sqrt{(1-9r^2)^2+(6\zeta r)^2}}\sin(31.416t-\phi_3) \qquad (E_2)$$

where
$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}(0.017226) = 0.0172 \text{ rad}$$
$$\phi_2 = \tan^{-1}\left(\frac{4\zeta r}{1-4r^2}\right) = \tan^{-1}(0.035229) = 0.0352 \text{ rad}$$
$$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1-9r^2}\right) = \tan^{-1}(0.054913) = 0.0549 \text{ rad}$$

Noting that $2\zeta r = 2(0.1)(0.0855) = 0.0171$
and $(1-r^2) = 0.9927$, Eq. (E_2) can be rewritten as

$$x_p(t) = [6.6389 - 13.7821\cos(10.472t-0.0172)$$
$$+ 15.7965\sin(10.472t-0.0172) + 2.2715\cos(20.944t-0.0352)$$
$$+ 8.4168\sin(20.944t-0.0352) + 2.6876\cos(31.416t-0.0549)$$
$$- 0.2896\sin(31.416t-0.0549)] \times 10^{-4} \text{ m.}$$

4.12 From problem 1.69,

$$F(t) = 1137.5 - 414.9436\cos 523.6t + 150.3139\sin 523.6t$$
$$+ 28.6058\cos 1047.2t - 146.1706\sin 1047.2t$$
$$+ 35.7278\cos 1570.8t + 55.1546\sin 1570.8t \qquad (E_1)$$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8000}{0.5}} = 126.4911$ rad/sec ; $\zeta = 0.06$

$r = \frac{\omega}{\omega_n} = \frac{523.6}{126.4911} = 4.1394$; $r^2 = 17.1348$; $1-r^2 = -16.1348$

$2\zeta r = 2(0.06)(4.1394) = 0.4967$

$$x_p(t) = \frac{1137.5}{k} - \frac{414.9436}{k} \frac{\cos(523.6t - \phi_1)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$+ \frac{150.3139}{k} \frac{\sin(523.6t - \phi_1)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} + \frac{28.6058}{k} \frac{\cos(1047.2t - \phi_2)}{\sqrt{(1-4r^2)^2 + (4\zeta r)^2}}$$

$$- \frac{146.1706}{k} \frac{\sin(1047.2t - \phi_2)}{\sqrt{(1-4r^2)^2 + (4\zeta r)^2}} + \frac{35.7278}{k} \frac{\cos(1570.8t - \phi_3)}{\sqrt{(1-9r^2)^2 + (6\zeta r)^2}}$$

$$+ \frac{55.1546}{k} \frac{\sin(1570.8t - \phi_3)}{\sqrt{(1-9r^2)^2 + (6\zeta r)^2}} \qquad (E_2)$$

Since $\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}(-0.0308) = -0.0308$

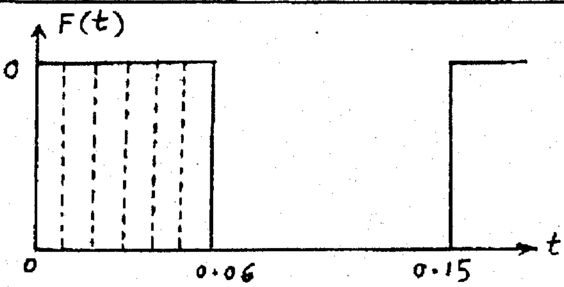
$$\phi_2 = \tan^{-1}\left(\frac{4\zeta r}{1-4r^2}\right) = \tan^{-1}(-0.0147) = -0.0147$$

$$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1-9r^2}\right) = \tan^{-1}(-0.00973) = -0.00973$$

the steady-state solution, Eq. (E_2), can be expressed as

$$x_p(t) = 0.1422 - 0.003211 \cos(523.6t + 0.0308)$$
$$+ 0.001163 \sin(523.6t + 0.0308) + 5.2921 \times 10^{-5} \cos(1047.2t + 0.0147)$$
$$- 2.7042 \times 10^{-4} \sin(1047.2t + 0.0147) + 2.9145 \times 10^{-5} \cos(1570.8t + 0.00973)$$
$$+ 4.4992 \times 10^{-5} \sin(1570.8t + 0.00973) \quad m$$

(4.13) $\omega_n = \sqrt{\frac{5 \times 10^6}{10 \times 10^3}} = 22.3607 \ \frac{rad}{sec}$



$\tau = 0.15$ sec

$\omega = \frac{2\pi}{\tau} = 41.888$ rad/sec

$r = \frac{\omega}{\omega_n} = 1.8733$, $r^2 = 3.5093$

$\zeta = 0$

Following data is used in Program # 1 to find Fourier coefficients in the expansion of $F(t)$:

t, sec	0.01	0.02	0.03	0.04	0.05	0.06	0.07	...	0.15
F(t), kN	400	400	400	400	400	400	0	...	0

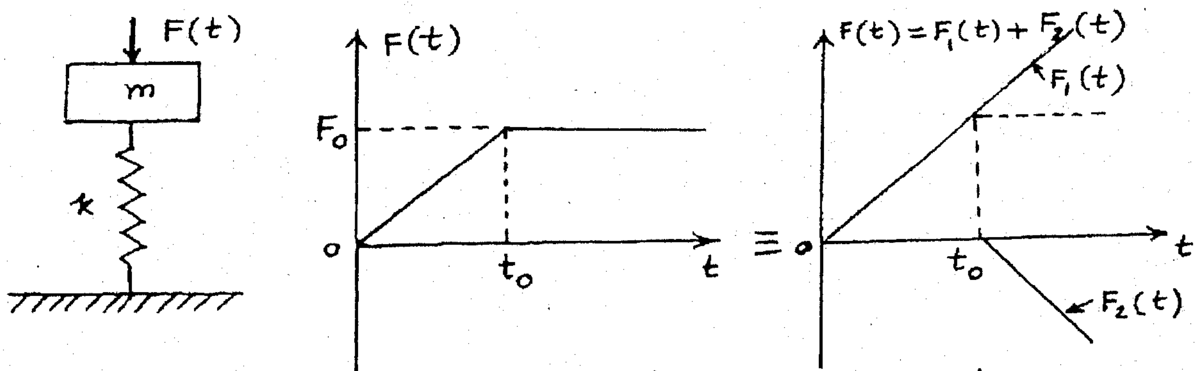
Result is:

$$F(t) = 160.0 + 25.5002 \cos 41.888t + 242.6276 \sin 41.888t$$
$$-75.3884 \cos 83.776t + 16.0237 \sin 83.776t$$
$$+16.4806 \cos 125.664t + 50.7237 \sin 125.664t$$
$$-62.3538 \cos 167.552t + 27.7604 \sin 167.552t$$
$$+ \cdots \quad kN$$

Since $\zeta = 0$ and all $\phi_j = 0$, $j = 1, 2, \cdots$, the steady-state response of the water tank, Eq. (4.13), becomes

$$x_p(t) = 0.032 + 2.0325 \times 10^{-3} \cos 41.888t$$
$$+ 19.3383 \times 10^{-3} \sin 41.888t - 1.1566 \times 10^{-3} \cos 83.776t$$
$$+ 0.2458 \times 10^{-3} \sin 83.776t + 0.1078 \times 10^{-3} \cos 125.664t$$
$$+ 0.3317 \times 10^{-3} \sin 125.664t - 0.2334 \times 10^{-3} \cos 167.552t$$
$$+ 0.1007 \times 10^{-3} \sin 167.552t + \cdots \quad m$$

4.14



Forcing function can be considered as the sum of two ramp functions, $F_1(t) = \dfrac{F_0\, t}{t_0}$ and $F_2(t) = -\dfrac{F_0 (t - t_0)}{t_0}$.

Response of the casting (undamped spring-mass system) to F_1 is given by

$$x_1(t) = \frac{F_0}{k}\left(\frac{t}{t_0} - \frac{\sin \omega_n t}{\omega_n t_0}\right) \quad \text{for } t \geq 0 \qquad (E_1)$$

Response due to F_2 can be obtained from Eq. (E_1) by replacing t by $t - t_0$ and F_0 by $-F_0$:

$$x_2(t) = -\frac{F_0}{k}\left\{\frac{t - t_0}{t_0} - \frac{\sin \omega_n (t - t_0)}{\omega_n t_0}\right\} \quad \text{for } t \geq t_0 \qquad (E_2)$$

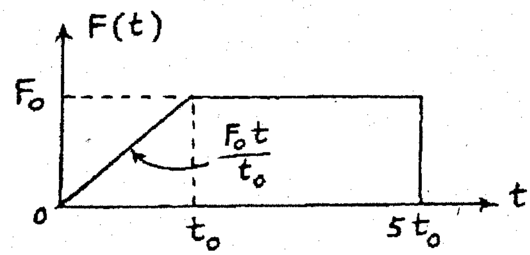
Total response of the casting is given by

$$x(t) = x_1(t) + x_2(t) = \frac{F_0}{k}\left\{1 + \frac{\sin\omega_n(t-t_0) - \sin\omega_n t}{\omega_n t_0}\right\}$$

for $t \geq t_0$ $\quad (E_3)$

4.15

$$F(t) = \begin{cases} (F_0 t/t_0) & ; \; 0 \le t \le t_0 \\ F_0 & ; \; t_0 \le t \le 5t_0 \\ 0 & ; \; t > 5t_0 \end{cases} \quad ---(E_1)$$



Response of the anvil is given by [Eq. (4.33) for an undamped system]:

$$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau \qquad (E_2)$$

For $0 \le t \le t_0$:

$$x(t) = \frac{F_0}{m\omega_n t_0}\int_0^t \tau\sin\omega_n(t-\tau)\,d\tau$$

$$= \frac{F_0}{k\,t_0}\left(t - \frac{1}{\omega_n}\sin\omega_n t\right) \qquad (E_3)$$

For $t_0 \le t \le 5t_0$:

$$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau$$

$$= \frac{F_0}{m\omega_n t_0}\int_0^{t_0}\tau\sin\omega_n(t-\tau)\,d\tau + \frac{F_0}{m\omega_n}\int_{t_0}^{t}\sin\omega_n(t-\tau)\,d\tau$$

$$= \frac{F_0}{m\omega_n t_0}\left[\sin\omega_n t\left\{\frac{1}{\omega_n^2}\cos\omega_n t_0 + \frac{t_0}{\omega_n}\sin\omega_n t_0 - \frac{1}{\omega_n^2}\right\}\right.$$

$$\left. - \cos\omega_n t\left\{\frac{1}{\omega_n^2}\sin\omega_n t_0 - \frac{t_0}{\omega_n}\cos\omega_n t_0\right\}\right]$$

$$+ \frac{F_0}{m\omega_n}\left\{\frac{1}{\omega_n}\cos(\omega_n t - \omega_n\tau)\right\}_{t_0}^{t}$$

$$= \frac{F_0}{k\,\omega_n t_0}\left[\sin\omega_n(t-t_0) - \sin\omega_n t + \omega_n t_0\right] \qquad (E_4)$$

For $t > 5t_0$:

$$x(t) = \frac{1}{m\omega_n}\left[\int_0^{t_0} \frac{F_0 \tau}{t_0} \sin \omega_n (t-\tau)\, d\tau + F_0 \int_{t_0}^{5t_0} \sin \omega_n (t-\tau)\, d\tau\right]$$

$$= \frac{F_0}{m\omega_n^2 t_0}\left[\frac{1}{\omega_n} \sin \omega_n (t-t_0) + t_0 \cos \omega_n (t-t_0) - \frac{1}{\omega_n} \sin \omega_n t\right]$$

$$+ \frac{F_0}{m\omega_n}\left[\frac{1}{\omega_n} \cos \omega_n (t-\tau)\right]_{t_0}^{5t_0}$$

$$= \frac{F_0}{k\omega_n t_0}\left[\sin \omega_n (t-t_0) - \sin \omega_n t + t_0 \omega_n \cos \omega_n (t-5t_0)\right] \quad (E_5)$$

4.16 From Eq.(4.33), $x(t) = \frac{1}{m\omega_d} \int_0^t F_0\, e^{-\alpha\tau}\, e^{-\zeta\omega_n (t-\tau)} \sin \omega_d (t-\tau)\, d\tau$

$$= \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t e^{-(\alpha-\zeta\omega_n)\tau} \sin \omega_d (t-\tau)\, d\tau$$

$$x(t) = \frac{F_0\, e^{-\zeta\omega_n t}}{m\omega_d} \int_0^t e^{-(\alpha-\zeta\omega_n)\tau} \cdot \left\{\sin \omega_d t \cdot \cos \omega_d \tau - \cos \omega_d t \cdot \sin \omega_d \tau\right\} d\tau$$

$$= \frac{F_0\, e^{-\zeta\omega_n t}}{m\omega_d} \sin \omega_d t \left[\frac{e^{-(\alpha-\zeta\omega_n)\tau}}{(\alpha-\zeta\omega_n)^2 + \omega_d^2}\left\{-(\alpha-\zeta\omega_n) \cos \omega_d \tau + \omega_d \sin \omega_d \tau\right\}\right]_0^t$$

$$- \frac{F_0\, e^{-\zeta\omega_n t}}{m\omega_d} \cos \omega_d t \left[\frac{e^{-(\alpha-\zeta\omega_n)\tau}}{(\alpha-\zeta\omega_n)^2 + \omega_d^2}\left\{-(\alpha-\zeta\omega_n) \sin \omega_d \tau - \omega_d \cos \omega_d \tau\right\}\right]_0^t$$

$$= \frac{F_0\, e^{-\zeta\omega_n t}}{m\omega_d\left[(\alpha-\zeta\omega_n)^2 + \omega_d^2\right]}\left\{\omega_d\, e^{-(\alpha-\zeta\omega_n)t} + (\alpha-\zeta\omega_n) \sin \omega_d t - \omega_d \cos \omega_d t\right\}$$

$$= \frac{F_0\, e^{-\alpha t}}{m\left[(\alpha-\zeta\omega_n)^2 + \omega_d^2\right]} + \frac{F_0\, e^{-\zeta\omega_n t}}{m\omega_d\left[(\alpha-\zeta\omega_n)^2 + \omega_d^2\right]} \sin(\omega_d t - \phi)$$

where $\phi = \tan^{-1}\left(\frac{\omega_d}{\alpha - \zeta\omega_n}\right)$.

4.17 Equation of motion:

$$m\ddot{x} + k x = A\, p(t)$$

or $10\ddot{x} + 1000\, x = \frac{\pi}{4} (0.1)^2 (50) (1 - e^{-3t}) = 0.3927 - 0.3927\, e^{-3t}$

Solution:

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{0.3927}{k} - \frac{0.3927}{k + m(3^2)} e^{-3t}$$

where $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10$ rad/sec. Thus x(t) becomes

$$x(t) = C_1 \cos 10\,t + C_2 \sin 10\,t + 39.27\,(10^{-5}) - 36.0275\,(10^{-5})\,e^{-3t}\ \text{m}$$

where C_1 and C_2 can be determined from the initial conditions. As $t \to \infty$, $e^{-3t} \to 0$ and the steady state response becomes

$$x(t) = C_1 \cos 10\,t + C_2 \sin 10\,t + 39.27\,(10^{-5})\ \text{m}$$

4.18 For $0 \le t \le \frac{\pi}{\omega}$: Equation of motion: $m\ddot{x} + k\,x = \frac{F_0}{2} - \frac{F_0}{2}\cos\omega t$

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{F_0}{2k} - \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}\cos\omega t$$

$$x(0)=0 \quad \therefore \quad A + \frac{F_0}{2k} - \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}, \quad A = \frac{F_0\left(\frac{\omega}{\omega_n}\right)^2}{2k\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}}$$

$$\dot{x}(0)=0 \quad \therefore \quad B = 0$$

$$x(t) = \frac{F_0}{2k\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}}\left[1 - \cos\omega t - \left(\frac{\omega}{\omega_n}\right)^2\left(1-\cos\omega_n t\right)\right]$$

At $t = \frac{\pi}{\omega}$, $$x\left(\frac{\pi}{\omega}\right) = \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}\left[2 - \frac{\omega^2}{\omega_n^2}\left(1-\cos\frac{\omega_n\pi}{\omega}\right)\right]$$

For $t > \frac{\pi}{\omega}$: Eq. (4.33) gives $x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau$

$$x(t) = \frac{1}{m\omega_n}\int_0^{\pi/\omega} F(\tau)\cdot\sin\omega_n(t-\tau)\,d\tau + \frac{1}{m\omega_n}\int_{\pi/\omega}^t F(\tau)\sin\omega_n(t-\tau)\,d\tau$$

$$= x\left(t=\frac{\pi}{\omega}\right) + \frac{F_0}{m\omega_n}\int_{\pi/\omega}^t \sin\omega_n(t-\tau)\,d\tau$$

$$= x\left(t=\frac{\pi}{\omega}\right) + \frac{F_0}{m\omega_n^2}\Big[\cos\omega_n(t-\tau)\Big]_{\tau=\pi/\omega}^{t}$$

$$= \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}\left[2 - \frac{\omega^2}{\omega_n^2}\left(1-\cos\frac{\omega_n\pi}{\omega}\right)\right] + \frac{F_0}{k}\left[1-\cos\omega_n\left(t-\frac{\pi}{\omega}\right)\right]$$

4.19 For an undamped system Eq. (4.33) gives

$$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau$$

$$F(\tau) = \begin{cases} F_0 & \text{for } 0 \le \tau \le t_0 \\ 0 & \text{for } \tau > t_0 \end{cases}$$

For $0 \le t \le t_0$: $\quad x(t) = \frac{F_0}{m\omega_n} \int_0^t \sin \omega_n (t-\tau)\, d\tau = \frac{F_0}{m\omega_n^2} (1 - \cos \omega_n t)$

$$= \frac{F_0}{k} (1 - \cos \omega_n t)$$

For $t > t_0$: $\quad x(t) = \frac{F_0}{m\omega_n} \int_0^{t_0} \sin \omega_n (t-\tau)\, d\tau = \frac{F_0}{m\omega_n} \left[\frac{1}{\omega_n} \cos \omega_n (t-\tau) \right]_{\tau=0}^{t_0}$

$$= \frac{F_0}{k} \left[\cos \omega_n (t - t_0) - \cos \omega_n t \right]$$

4.20 $F(\tau) = \begin{cases} F_0 \left(\frac{\tau}{t_0} \right) & \text{for } 0 \le \tau \le t_0 \\ 0 & \text{for } \tau > t_0 \end{cases}$

For $0 \le t \le t_0$: $\quad x(t) = \frac{F_0}{m\omega_n t_0} \int_0^t \tau \sin \omega_n (t-\tau)\, d\tau$

i.e. $\quad x(t) = \frac{F_0}{m\omega_n t_0} \left[\int_0^t (t-\tau) \sin \omega_n (t-\tau) (-d\tau) - t \int_0^t \sin \omega_n (t-\tau) (-d\tau) \right]$

$$= \frac{F_0}{m\omega_n t_0} \left[\frac{1}{\omega_n^2} \sin \omega_n (t-\tau) - \frac{(t-\tau)}{\omega_n} \cos \omega_n (t-\tau) \right]_{\tau=0}^{t}$$

$$+ \frac{F_0\, t}{m\omega_n t_0} \left[\frac{\cos \omega_n (t-\tau)}{\omega_n} \right]_{\tau=0}^{t}$$

$$= \frac{F_0}{k\, t_0} \left(t - \frac{1}{\omega_n} \sin \omega_n t \right)$$

For $t > t_0$:

$$x(t) = \frac{F_0}{m\omega_n t_0} \int_0^{t_0} \tau \sin \omega_n (t-\tau)\, d\tau$$

$$= \frac{F_0}{m\omega_n t_0} \left[\frac{1}{\omega_n^2} \sin \omega_n (t-\tau) - \frac{(t-\tau)}{\omega_n} \cos \omega_n (t-\tau) \right]_{\tau=0}^{t}$$

$$+ \frac{F_0\, t}{m\omega_n t_0} \left[\frac{\cos \omega_n (t-\tau)}{\omega_n} \right]_{\tau=0}^{t}$$

$$= \frac{F_0}{k\, t_0} \left[\frac{1}{\omega_n} \sin \omega_n (t - t_0) + t_0 \cos \omega_n (t - t_0) - \frac{1}{\omega_n} \sin \omega_n t \right]$$

4.21 $F(t) = \begin{cases} F_0 \left(1 - \cos \frac{\pi t}{2 t_0} \right) & ; \ 0 \le t \le t_0 \\ 0 & ; \ t > t_0 \end{cases} \qquad (E_1)$

For an undamped system, Eq. (4.33) gives

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n (t-\tau)\, d\tau$$

For $0 \le t \le t_0$:

$$x(t) = \frac{F_0}{m\,\omega_n} \int_0^t \left(1 - \cos\frac{\pi\tau}{2t_0}\right) \sin\omega_n(t-\tau)\, d\tau \qquad (E_2)$$

Noting that

$$\int_0^t \sin(\omega_n t - \omega_n\tau)\, d\tau = \left[\frac{1}{\omega_n}\cos(\omega_n t - \omega_n \tau)\right]_0^t = \frac{1}{\omega_n}(1 - \cos\omega_n t)$$

and

$$\int_0^t \cos\frac{\pi\tau}{2t_0}\left(\sin\omega_n t\, \cos\omega_n\tau - \cos\omega_n t\, \sin\omega_n\tau\right) d\tau$$

$$= \sin\omega_n t \left[\frac{\sin\left(\frac{\pi}{2t_0} - \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} + \frac{\sin\left(\frac{\pi}{2t_0} + \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} + \omega_n\right)}\right]_0^t$$

$$- \cos\omega_n t \left[-\frac{\cos\left(\omega_n - \frac{\pi}{2t_0}\right)\tau}{2\left(\omega_n - \frac{\pi}{2t_0}\right)} - \frac{\cos\left(\omega_n + \frac{\pi}{2t_0}\right)\tau}{2\left(\omega_n + \frac{\pi}{2t_0}\right)}\right]_0^t$$

$$= \frac{-1}{2\left(\frac{\pi}{2t_0} - \omega_n\right)}\left[\cos\frac{\pi t}{2t_0} - \cos\omega_n t\right] + \frac{1}{2\left(\frac{\pi}{2t_0} + \omega_n\right)}\left[\cos\frac{\pi t}{2t_0} - \cos\omega_n t\right],$$

Eq. (E_2) can be simplified as

$$x(t) = \frac{F_0}{k}(1 - \cos\omega_n t) + \frac{(F_0/m)}{\left\{\left(\frac{\pi}{2t_0}\right)^2 - \omega_n^2\right\}}\left[\cos\frac{\pi t}{2t_0} - \cos\omega_n t\right] \qquad (E_3)$$

For $t > t_0$:

$$x(t) = \frac{F_0}{m\,\omega_n}\int_0^{t_0}\left(1 - \cos\frac{\pi\tau}{2t_0}\right)\sin\omega_n(t-\tau)\, d\tau \qquad (E_4)$$

$$= \frac{F_0}{m\,\omega_n}\left[\frac{1}{\omega_n}\cos\omega_n(t-t_0) - \frac{1}{\omega_n}\cos\omega_n t\right] - \frac{F_0}{m\,\omega_n}\left\{\sin\omega_n t \,*\right.$$

$$\left[\frac{\sin\left(\frac{\pi}{2t_0} - \omega_n\right)t_0}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} + \frac{\sin\left(\frac{\pi}{2t_0} + \omega_n\right)t_0}{2\left(\frac{\pi}{2t_0} + \omega_n\right)}\right] + \cos\omega_n t \,*$$

$$\left.\left[\frac{\cos\left(\frac{\pi}{2t_0} - \omega_n\right)t_0}{2\left(\omega_n - \frac{\pi}{2t_0}\right)} + \frac{\cos\left(\frac{\pi}{2t_0} + \omega_n\right)t_0}{2\left(\omega_n + \frac{\pi}{2t_0}\right)} - \frac{1}{2\left(\omega_n - \frac{\pi}{2t_0}\right)} - \frac{1}{2\left(\omega_n + \frac{\pi}{2t_0}\right)}\right]\right\}$$

i.e., $$x(t) = \frac{F_0}{k}\left[\cos\omega_n(t - t_0) - \cos\omega_n t\right]$$

$$- \frac{F_o}{2m\omega_n\left(\frac{\pi}{2t_o} - \omega_n\right)} \sin \omega_n (t - t_o) - \frac{F_o}{2m\omega_n\left(\frac{\pi}{2t_o} + \omega_n\right)} \sin \omega_n (t - t_o)$$

$$+ \frac{F_o}{2m\omega_n} \cos \omega_n t \cdot \left\{ - \frac{1}{\left(\frac{\pi}{2t_o} - \omega_n\right)} + \frac{1}{\left(\frac{\pi}{2t_o} + \omega_n\right)} \right\} \qquad (E_5)$$

4.22 Base displacement $= y(s) = Y \sin \frac{\pi s}{\delta}$ $\qquad (E_1)$

i.e., $y(t) = Y \sin \frac{\pi t}{t_o}$ $\qquad (E_2)$

Steady-state relative displacement can be found from Eq. (4.36) as

$$z(t) = -\frac{1}{\omega_d} \int_0^t \ddot{y}(\tau)\, e^{-\zeta\omega_n (t-\tau)} \sin \omega_d (t-\tau)\, d\tau \qquad (E_3)$$

where $\quad \ddot{y}(\tau) = -Y\left(\frac{\pi}{t_o}\right)^2 \sin \frac{\pi\tau}{t_o}$ $\qquad (E_4)$

$$z(t) = \frac{Y}{\omega_d}\left(\frac{\pi}{t_o}\right)^2 \int_0^t e^{-\zeta\omega_n t}\, e^{\zeta\omega_n \tau} \sin \frac{\pi\tau}{t_o} \sin \omega_d (t-\tau)\, d\tau \qquad (E_5)$$

But $\quad \sin \frac{\pi\tau}{t_o} \sin \omega_d (t-\tau) = \frac{1}{2}\cos\left(\frac{\pi\tau}{t_o} - \omega_d t + \omega_d \tau\right) - \frac{1}{2}\cos\left(\frac{\pi\tau}{t_o} + \omega_d t - \omega_d \tau\right)$

$$= \frac{1}{2}\left[\cos\left(\frac{\pi}{t_o} + \omega_d\right)\tau \cdot \cos \omega_d t + \sin\left(\frac{\pi}{t_o} + \omega_d\right)\tau \cdot \sin \omega_d t\right]$$

$$-\frac{1}{2}\left[\cos\left(\frac{\pi}{t_o} - \omega_d\right)\tau \cdot \cos \omega_d t - \sin\left(\frac{\pi}{t_o} - \omega_d\right)\tau \cdot \sin \omega_d t\right] \qquad (E_6)$$

Eqs. (E_5) and (E_6) give

$$z(t) = \frac{1}{2}\frac{Y}{\omega_d}\left(\frac{\pi}{t_o}\right)^2 e^{-\zeta\omega_n t}\left[\cos \omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cos\left(\frac{\pi}{t_o} + \omega_d\right)\tau \cdot d\tau\right.$$

$$+ \sin \omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cdot \sin\left(\frac{\pi}{t_o} + \omega_d\right)\tau \cdot d\tau$$

$$- \cos \omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cdot \cos\left(\frac{\pi}{t_o} - \omega_d\right)\tau \cdot d\tau$$

$$\left. + \sin \omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cdot \sin\left(\frac{\pi}{t_o} - \omega_d\right)\tau \cdot d\tau\right] \qquad (E_7)$$

Eq. (E_7) can be simplified

$$z(t) = \frac{Y}{2\omega_d}\left(\frac{\pi}{t_o}\right)^2 \frac{1}{(\zeta\omega_n)^2 + \left(\frac{\pi}{t_o} + \omega_d\right)^2}\left\{\zeta\omega_n \cos \frac{\pi t}{t_o}\right.$$

$$+ \left(\frac{\pi}{t_o} + \omega_d\right) \sin \frac{\pi t}{t_o} - \zeta\omega_n\, e^{-\zeta\omega_n t} \cos \omega_d t$$

$$\left. + \left(\frac{\pi}{t_o} + \omega_d\right) e^{-\zeta\omega_n t} \sin \omega_d t\right\}$$

$$+ \frac{Y}{2\omega_d}\left(\frac{\pi}{t_o}\right)^2 \frac{1}{(\zeta\omega_n)^2 + \left(\frac{\pi}{t_o} - \omega_d\right)^2}\left\{-\zeta\omega_n \cos \frac{\pi t}{t_o}\right.$$

$$-\left(\frac{\pi}{t_0}-\omega_d\right)\sin\frac{\pi t}{t_0} + \zeta\omega_n\, e^{-\zeta\omega_n t}\cos\omega_d t$$
$$+\left(\frac{\pi}{t_0}-\omega_d\right) e^{-\zeta\omega_n t}\sin\omega_d t \Big\} \qquad (E_8)$$

(4.23) Base displacement:

$$y(t)=\begin{cases}\dfrac{Y\,v\,t}{\delta}\ ; & 0\le t\le t_0=\dfrac{\delta}{v}\\ 0\ ; & t>t_0=\dfrac{\delta}{v}\end{cases} \qquad (1)$$

Equation of motion of vehicle:

$$m\ddot{x}+k\,(x-y)=0 \qquad (2)$$

Using Eq. (1), Eq. (2) can be expressed as

$$m\ddot{x}+k\,x=k\,y=\begin{cases}\dfrac{k\,v\,Y\,t}{\delta}\ ; & 0\le t\le t_0\\ 0\ ; & t>t_0\end{cases} \qquad (3)$$

Steady state solution of Eq. (3) from Example 4.6:

$$x(t)=\begin{cases}\dfrac{v\,Y}{\delta\,\omega_n\,k}\left\{\omega_n\,t-\sin\omega_n\,t\right\}\ ; & 0\le t\le t_0\\ 0\ ; & t>t_0\end{cases} \qquad (4)$$

Note that the homogeneous solution,

$$x(t)=C_1\cos\omega_n\,t+C_2\sin\omega_n\,t \qquad (5)$$

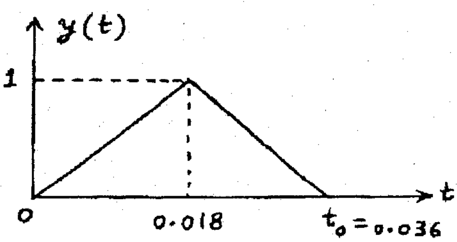
is to be added to Eq. (4) to obtain the complete solution. The constants C_1 and C_2 are to be evaluated from the initial conditions (at $t=0$). In fact, tthe resulting complete solution is valid for all values of t, including values of $t>t_0$.

(4.24) Speed of automobile = 50 km/hr

Excitation frequency = $\left(\frac{50\times1000}{3600}\right)\frac{1}{0.5}=27.7778$ Hz

Natural frequency = $f_n=1.0$ Hz $\Rightarrow$ $\omega_n=2\pi$ rad/sec

$$t_0=\frac{0.5\times3600}{50\times1000}=0.036\ \text{sec}$$



$$y(t)=\begin{cases}\dfrac{0.2\,t}{t_0}\ ; & 0\le t\le t_0/2\\ -\dfrac{0.2\,t}{t_0}+0.2\ ; & \dfrac{t_0}{2}\le t\le t_0\\ 0\ ; & t>t_0\end{cases} \quad \cdots\ (E_1)$$

Equation of motion (for undamped case):

$$m\ddot{x} + k(x-y) = 0 \qquad \text{or} \qquad m\ddot{x} + kx = ky = F(t) \tag{E$_2$}$$

$$\text{where} \qquad F(t) = k\,y(t) \tag{E$_3$}$$

Solution of Eq. (E_2) is: $$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau \tag{E$_4$}$$

For $0 \le t \le \frac{t_0}{2}$:

$$x(t) = \frac{k}{m\omega_n}\int_0^t \left(\frac{0.2}{t_0}\right)\tau\sin\omega_n(t-\tau)\,d\tau \tag{E$_5$}$$

Since $$\int_0^t \tau\sin\omega_n(t-\tau)\,d\tau = \left(\frac{t}{\omega_n} - \frac{1}{\omega_n^2}\sin\omega_n t\right),$$

Eq. (E_5) becomes

$$x(t) = 5.5556\,(t - 0.1592\sin 6.2832\,t) \text{ m} \quad ; 0 \le t \le 0.018 \text{ sec} \tag{E$_6$}$$

For $\frac{t_0}{2} \le t \le t_0$:

$$x(t) = \frac{k}{m\omega_n}\left\{\int_0^{t_0/2}\frac{0.2\tau}{t_0}\sin\omega_n(t-\tau)\,d\tau + \int_{t_0/2}^{t}\left(-\frac{0.2\tau}{t_0}+0.2\right)\sin\omega_n(t-\tau)\,d\tau\right\}$$

But

$$\frac{0.2k}{m\omega_n t_0}\cdot\int_0^{t_0/2}\tau\cdot\sin\omega_n(t-\tau)\,d\tau = \frac{0.2k}{m\omega_n t_0}\left\{\sin\omega_n t\left[\frac{1}{\omega_n^2}\cos\omega_n\tau + \frac{\tau}{\omega_n}\sin\omega_n\tau\right]_0^{t_0/2} - \cos\omega_n t\left[\frac{1}{\omega_n^2}\sin\omega_n\tau - \frac{\tau}{\omega_n}\cos\omega_n\tau\right]_0^{t_0/2}\right\}$$

$$= 5.5556\left[0.1592\sin 6.2832(t-0.018) + 0.0180\cos 6.2832(t-0.018) - 0.1592\sin 6.2832\,t\right] \text{ m} \tag{E$_7$}$$

since $t_0 = 0.036$.

$$-\frac{0.2k}{m\omega_n t_0}\int_{t_0/2}^{t}\tau\sin\omega_n(t-\tau)\,d\tau = -\frac{0.2k}{m\omega_n t_0}\left\{\sin\omega_n t\left[\frac{1}{\omega_n^2}\cos\omega_n\tau + \frac{\tau}{\omega_n}\sin\omega_n\tau\right]_{t_0/2}^{t} - \cos\omega_n t\cdot\left[\frac{1}{\omega_n^2}\sin\omega_n\tau - \frac{\tau}{\omega_n}\cos\omega_n\tau\right]_{t_0/2}^{t}\right\}$$

$$= -5.5556\left[t - 0.1592\sin 6.2832(t-0.018) - 0.0180\cos 6.2832(t-0.018)\right] \text{ m} \tag{E$_8$}$$

$$\frac{0.2\,k}{m\,\omega_n}\int_{t_0/2}^{t}\sin\omega_n(t-\tau)\,d\tau = \frac{0.2\,k}{m\,\omega_n}\left[\frac{1}{\omega_n}\cos\omega_n(t-\tau)\right]_{t_0/2}^{t}$$

$$= \frac{0.2\,k}{m\,\omega_n^2}\left[1-\cos\omega_n\left(t-\frac{t_0}{2}\right)\right]$$

$$= 0.2\left[1-\cos 6.2832\,(t-0.018)\right]\ \text{m} \qquad (E_9)$$

Hence the solution can be expressed as

$$x(t) = \left[1.7689\sin 6.2832\,(t-0.018) - 0.8845\sin 6.2832\,t - 5.5556\,t + 0.2\right]\ \text{m}\ ; \quad 0.018 \le t \le 0.036\ \text{sec} \qquad (E_{10})$$

For $t > t_0$:

$$x(t) = \frac{0.2\,k}{m\,\omega_n t_0}\int_0^{t_0/2}\tau\sin\omega_n(t-\tau)\,d\tau - \frac{0.2\,k}{m\,\omega_n t_0}\int_{t_0/2}^{t_0}\tau\sin\omega_n(t-\tau)\,d\tau + \frac{0.2\,k}{m\,\omega_n}\int_{t_0/2}^{t_0}\sin\omega_n(t-\tau)\,d\tau \qquad (E_{11})$$

The first term on the right side of (E_{11}) is given by (E_7).

Second term on the right side of (E_{11}) is

$$-\frac{0.2\,k}{m\,\omega_n t_0}\int_{t_0/2}^{t_0}\tau\sin\omega_n(t-\tau)\,d\tau = -\frac{0.2\,k}{m\,\omega_n t_0}\left\{\sin\omega_n t\cdot\left[\frac{1}{\omega_n^2}\cos\omega_n\tau + \frac{\tau}{\omega_n}\sin\omega_n\tau\right]_{t_0/2}^{t_0} - \cos\omega_n t\cdot\left[\frac{1}{\omega_n^2}\sin\omega_n\tau - \frac{\tau}{\omega_n}\cos\omega_n\tau\right]_{t_0/2}^{t_0}\right\}$$

$$= -5.5556\left[0.1592\sin 6.2832\,(t-0.036) + 0.0360\cos 6.2832\,(t-0.036) - 0.1592\sin 6.2832\,(t-0.018) - 0.0180\cos 6.2832\,(t-0.018)\right]\ \text{m} \qquad (E_{12})$$

The third term on the right side of (E_{11}) is:

$$\frac{0.2\,k}{m\,\omega_n}\int_{t_0/2}^{t_0}\sin\omega_n(t-\tau)\,d\tau = \frac{0.2\,k}{m\,\omega_n}\left[\frac{1}{\omega_n}\cos\omega_n(t-\tau)\right]_{t_0/2}^{t_0}$$

$$= \frac{0.2\,k}{m\,\omega_n^2}\left[\cos\omega_n(t-t_0) - \cos\omega_n\left(t-\frac{t_0}{2}\right)\right]$$

$$= 0.2\left[\cos 6.2832\,(t-0.036) - \cos 6.2832\,(t-0.018)\right]\ \text{m} \qquad (E_{13})$$

$\therefore$ $x(t)$ is given by the sum of Eqs. (E_7), (E_{12}) and (E_{13}), which can be simplified as

$$x(t) = 1.7689 \sin 6.2832\,(t - 0.018) - 0.8845 \sin 6.2832\, t$$
$$- 0.8845 \sin 6.2832\,(t - 0.036) \quad m \;; \quad t > 0.036 \text{ sec} \qquad (E_{14})$$

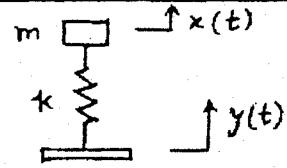
4.25 When the container strikes the floor, the velocity of the mass is given by

$$m g h = \tfrac{1}{2} m v^2 \quad \text{or} \quad v = \sqrt{2 g h} \qquad (E_1)$$

The displacement of the camcorder subjected to an initial velocity $\dot{x}_0 = v$ is given by Eq. (2.72), with $x_0 = 0$ and $\zeta < 1$,

$$x(t) = e^{-\zeta \omega_n t} \cdot \frac{\dot{x}_0}{\omega_n \sqrt{1-\zeta^2}} \cdot \sin \sqrt{1-\zeta^2}\; \omega_n t \qquad (E_2)$$

4.26 System can be modeled as a spring-mass system subjected to base motion:



$$y(t) = \begin{cases} (Y t^2 / t_0^2) \;; & 0 \le t \le t_0 \\ 0 \;; & t > t_0 \end{cases} \qquad --- \; (E_1)$$

Relative displacement of mass, $z = x - y$, is given by Eq. (4.36):

$$z(t) = -\frac{1}{\omega_d} \int_0^t \ddot{y}(\tau)\, e^{-\zeta \omega_n (t-\tau)} \sin \omega_d (t-\tau)\, d\tau \qquad (E_2)$$

where

$$\left.\begin{aligned} y(\tau) &= (Y \tau^2 / t_0^2) \\ \ddot{y}(\tau) &= (2Y/t_0^2) \end{aligned}\right\} \;; \quad 0 \le \tau \le t_0 \qquad (E_3)$$

and $\qquad y(\tau) = 0 \qquad ; \quad \tau > t_0$

For $0 \le t \le t_0$:

Since the system is undamped, $\omega_d = \omega_n$ and $\zeta = 0$, and (E_2) reduces to

$$z(t) = -\frac{2Y}{\omega_n t_0^2} \int_0^t \sin \omega_n (t-\tau)\, d\tau \qquad (E_4)$$

Here

$$\int_0^t \sin \omega_n (t-\tau)\, d\tau = \int_0^t (\sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau)\, d\tau$$
$$= \sin \omega_n t \int_0^t \cos \omega_n \tau\, d\tau - \cos \omega_n t \int_0^t \sin \omega_n \tau\, d\tau$$

$$= \sin\omega_n t\left(\frac{1}{\omega_n}\sin\omega_n\tau\right)_0^t - \cos\omega_n t\left(-\frac{1}{\omega_n}\cos\omega_n\tau\right)_0^t$$

$$= \left(\frac{1-\cos\omega_n t}{\omega_n}\right) \qquad (E_5)$$

Thus Eq. (E_4) gives

$$x(t) = \frac{Y\,t^2}{t_0^2} - \frac{2Y}{t_0^2\,\omega_n^2}\left(1-\cos\omega_n t\right) \quad ; \quad 0 \le t \le t_0 \qquad (E_6)$$

For $t > t_0$:

Eq. (E_2) gives

$$z(t) = -\frac{1}{\omega_n}\int_0^{t_0}\frac{2Y}{t_0^2}\sin\omega_n(t-\tau)\,d\tau \qquad (E_7)$$

But

$$\int_0^{t_0}\sin\omega_n(t-\tau)\,d\tau = \frac{1}{\omega_n}\left\{\cos\omega_n(t-t_0) - \cos\omega_n t\right\}$$

$$\therefore\ z(t) = x(t) - y(t) = -\frac{2Y}{\omega_n^2 t_0^2}\left\{\cos\omega_n(t-t_0) - \cos\omega_n t\right\} \ ; \ t > t_0 \qquad (E_8)$$

4.27 $$I_1 = \int_0^t (t-\tau)\,e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)(-d\tau)$$

$$= \left[\frac{(t-\tau)\,e^{-\zeta\omega_n(t-\tau)}}{(\zeta\omega_n)^2+\omega_d^2}\left\{-\zeta\omega_n\sin\omega_d(t-\tau) - \omega_d\cos\omega_d(t-\tau)\right\}\right.$$

$$\left. - \frac{e^{-\zeta\omega_n(t-\tau)}}{(\zeta\omega_n)^2+\omega_d^2}\left\{\left(\zeta^2\omega_n^2-\omega_d^2\right)\sin\omega_d(t-\tau) + 2\zeta\omega_n\omega_d\cos\omega_d(t-\tau)\right\}\right]_{\tau=0}^{t}$$

$$= -\frac{1}{\omega_n^4}(2\zeta\omega_n\omega_d) + \frac{t\,e^{-\zeta\omega_n t}}{\omega_n^2}\left(\zeta\omega_n\sin\omega_d t + \omega_d\cos\omega_d t\right)$$

$$+ \frac{e^{-\zeta\omega_n t}}{\omega_n^4}\left\{\omega_n^2(2\zeta^2-1)\sin\omega_d t + 2\zeta\omega_n\omega_d\cos\omega_d t\right\}$$

$$I_2 = \int_0^t e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)(-d\tau)$$

$$= \left[\frac{e^{-\zeta\omega_n(t-\tau)}}{\zeta^2\omega_n^2+\omega_d^2}\left\{-\zeta\omega_n\sin\omega_d(t-\tau) - \omega_d\cos\omega_d(t-\tau)\right\}\right]_{\tau=0}^{t}$$

$$= -\frac{\omega_d}{\omega_n^2} + \frac{e^{-\zeta\omega_n t}}{\omega_n^2}\left(\zeta\omega_n\sin\omega_d t + \omega_d\cos\omega_d t\right)$$

$$x(t) = \frac{\delta F}{m\,\omega_d} \cdot I_1 - \frac{\delta F \cdot t}{m\,\omega_d} \cdot I_2$$

$$= \frac{\delta F}{k}\left[t - \frac{2\zeta}{\omega_n} + e^{-\zeta\omega_n t}\left\{\frac{2\zeta}{\omega_n}\cos\omega_d t - \left(\frac{\omega_d^2-\omega_n^2}{\omega_n^2\omega_d}\right)\sin\omega_d t\right\}\right]$$

4.28 (1) $m\ddot{x} + c\dot{x} + k x = F(t)$

where $m(t) = M - m_0 t = (2000 - 10\,t)$ kg

and $F(t) = m_0 v = 10\,(2000) = 20000$ N

(2) With $m = M - \frac{1}{2} m_0 t_0 = 2000 - \frac{1}{2}(10)(100) = 1500$ kg, equation of motion becomes

$$1500\,\ddot{x} + 0.1 \times 10^6\,\dot{x} + 7.5 \times 10^6\,x = 20\,000 = \text{constant}$$

Maximum steady state displacement is

$$x_p(t) = \frac{F}{k} = \frac{20000}{7.5 \times 10^6} = 0.002667 \text{ m}$$

4.29 From Eq. (4.32), the response to unit step function can be obtained by setting $F(\tau) = 1$ as

$$h(t) = \int_0^t g(t-\tau)\,d\tau \qquad \text{---- } (E_1)$$

By differentiating this equation with respect to t, we obtain

$$\frac{dh}{dt}(t) = g(t)$$

4.30 Equation (4.32) gives $\quad x(t) = \int_0^t F(\tau)\cdot g(t-\tau)\cdot d\tau$

But $g(t-\tau) = \frac{dh}{d\tau}(t-\tau)$ from problem 4-29.

$$x(t) = \int_0^t F(\tau)\,\frac{dh}{d\tau}(t-\tau)\,d\tau$$

Integration by parts gives

$$x(t) = -F(t)\cdot h(t-\tau)\Big|_{\tau=0}^{t} + \int_0^t \frac{dF}{d\tau}\cdot h(t-\tau)\,d\tau$$

$$= -F(t)\,h(0) + F(0)\,h(t) + \int_0^t \frac{dF}{d\tau}\,h(t-\tau)\,d\tau$$

But $h(0) = 0$ from Eq. (E_1) of problem 4.29.

$$\therefore\ x(t) = F(0)\,h(t) + \int_0^t \frac{dF}{d\tau}(\tau)\cdot h(t-\tau)\cdot d\tau$$

4.31 Equation of motion for rotation about O:

$$J_0\,\ddot{\theta} + M\,\ddot{x}\,(\ell) + k_1\,a^2\,\theta + k_2\,b^2\,\theta = F_0\,\ell\,e^{-t} \tag{1}$$

where $\ddot{x} = \ell\,\ddot{\theta}$ and

$$J_0 = \frac{1}{12}\,m\,\ell^2 + m\left(\frac{\ell}{2}\right)^2 = \frac{1}{3}\,m\,\ell^2$$

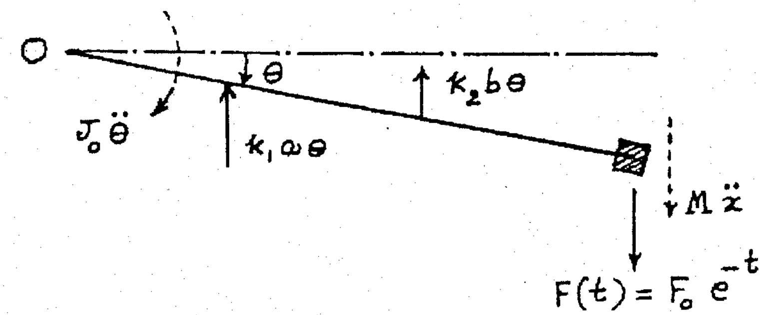
Eq. (1) can be rewritten as:

$$\left(\frac{1}{3}\, m\, \ell^2 + M\, \ell^2\right) \ddot{\theta} + (k_1\, a^2 + k_2\, b^2)\, \theta = F_0\, \ell\, e^{-t} \tag{2}$$

For given data, Eq. (2) takes the form:

$$53.3333\, \ddot{\theta} + 1562.5\, \theta = 500\, e^{-t} \tag{3}$$

Noting that the system is undamped with $\omega_n = \sqrt{\dfrac{1562.5}{53.3333}} = 5.4127$ rad/sec and



the forcing term as $500\, e^{-t}$, the convolution integral, Eq. (4.33), can be used to find the steady state response as:

$$\begin{aligned}\theta(t) &= \frac{1}{(53.3333)\,(5.4127)} \int_0^t 500\, e^{-\tau} \sin 5.4127\,(t-\tau)\, d\tau \\ &= 1.7320 \int_0^t e^{-t}\, e^{(t-\tau)} \sin 5.4127\,(t-\tau)\, d\tau \\ &= -\,1.7320\, e^{-t} \int_0^t e^{(t-\tau)} \sin 5.4127\,(t-\tau)\,(-\,d\tau)\end{aligned} \tag{4}$$

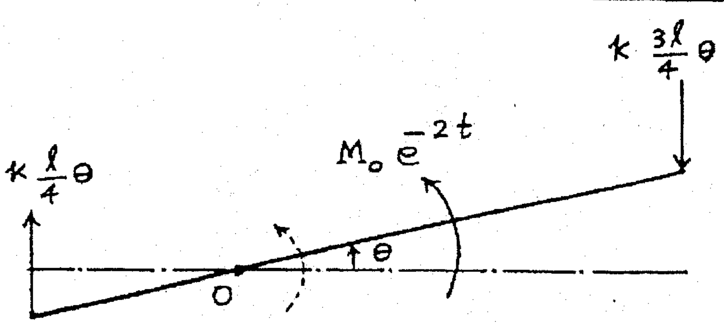
Using the formula:

$$\int e^{a\,x} \sin b\,x\, dx = \frac{1}{a^2 + b^2}\, e^{a\,x} \left[a \sin b\,x - b \cos b\,x\right] \tag{5}$$

Eq. (4) can be expressed as

$$\begin{aligned}\theta(t) &= -\,1.7320\, e^{-t} \left[\frac{e^{(t-\tau)}}{1^2 + 5.4127^2} \left\{\sin 5.4127\,(t-\tau) - 5.4127 \cos 5.4127\,(t-\tau)\right\}\right]_{\tau=0}^{\tau=t} \\ &= 0.3094\, e^{-t} + 0.05717 \sin 5.4127\, t - 0.3094 \cos 5.4127\, t \text{ radian}\end{aligned}$$

4.32



$$J_0 = \frac{1}{12}\, m\, \ell^2 + m \left(\frac{\ell}{4}\right)^2 = \frac{7}{48}\, m\, \ell^2 = \frac{7}{48}\,(10)\,(1^2) = 1.4583\ \text{kg–m}^2$$

Equation of motion for rotation about O:

$$J_0\, \ddot{\theta} + k\, \frac{\ell^2}{16}\, \theta + k\, \frac{9}{16}\, \ell^2\, \theta = M_0\, e^{-2t}$$

$$\text{or}\quad J_0\, \ddot{\theta} + \frac{5}{8}\, k\, \ell^2\, \theta = M_0\, e^{-2t}$$

$$\text{or}\quad 1.4583\, \ddot{\theta} + 3125.0\, \theta = 100\, e^{-2t} \tag{1}$$

Noting that the system is undamped with $\omega_n = \sqrt{\frac{3125.0}{1.4583}} = 46.2915$ rad/sec, the convolution integral, Eq. (4.33), can be used to find the steady state response as:

$$\theta(t) = \frac{1}{(1.4583)\,(46.2915)} \int_0^t (100\, e^{-2\tau})\, \sin 46.2915\,(t-\tau)\, d\tau$$

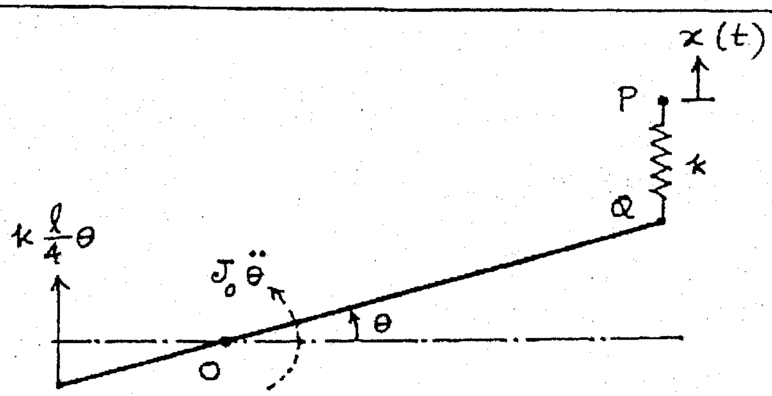
$$= -1.4813\, e^{-2t} \int_0^t e^{2(t-\tau)}\, \sin 46.2915\,(t-\tau)\,(-d\tau) \tag{2}$$

Using Eq. (5) in the solution of Problem 4.31, Eq. (2) can be expressed as:

$$\theta(t) = -1.4813\, e^{-2t} \left[\frac{e^{2(t-\tau)}}{2^2 + 46.2915^2} \left\{ 2 \sin 46.2915\,(t-\tau) - 46.2915 \cos 46.2915\,(t-\tau) \right\} \right]_{\tau=0}^{\tau=t}$$

$$= 0.03194\, e^{-2t} + 13.7994\,(10^{-4})\, \sin 46.2915\, t - 0.03194 \cos 46.2915\, t \ \text{radian} \tag{4}$$

4.33



Net compression of spring PQ $= \frac{3\,\ell\,\theta}{4} - x(t)$. Equation of motion for rotation about O:

$$J_0\, \ddot{\theta} = -k\, \frac{\ell\,\theta}{4} \cdot \left(\frac{\ell}{4}\right) - k \left(\frac{3\,\ell\,\theta}{4} - x(t)\right) \left(\frac{3\,\ell}{4}\right)$$

$$\text{or}\quad J_0\, \ddot{\theta} + \frac{5}{8}\, k\, \ell^2\, \theta = \frac{3\,k\,\ell}{4}\, x(t) = \frac{3\,k\,\ell}{4}\, x_0\, e^{-t} \tag{1}$$

For given data, $J_0 = \frac{1}{12}\, m\, \ell^2 + m\,(\frac{\ell}{4})^2 = \frac{7}{48}\, m\, \ell^2 = \frac{7}{48}\,(10)\,(1^2) = 1.4583$ kg–m^2, $\frac{5}{8}\, k\, \ell^2 = \frac{5}{8}\,(5000)\,(1^2) = 3125$ N/m, and Eq. (1) becomes:

$$1.4583\, \ddot{\theta} + 3125.0\, \theta = 37.5\, e^{-t} \tag{2}$$

Noting that the system is undamped with

$$\omega_n = \sqrt{\frac{3125.0}{1.4583}} = 46.2915 \text{ rad/sec}$$

the convolution integral, Eq. (4.33), can be used to find the steady state response as:

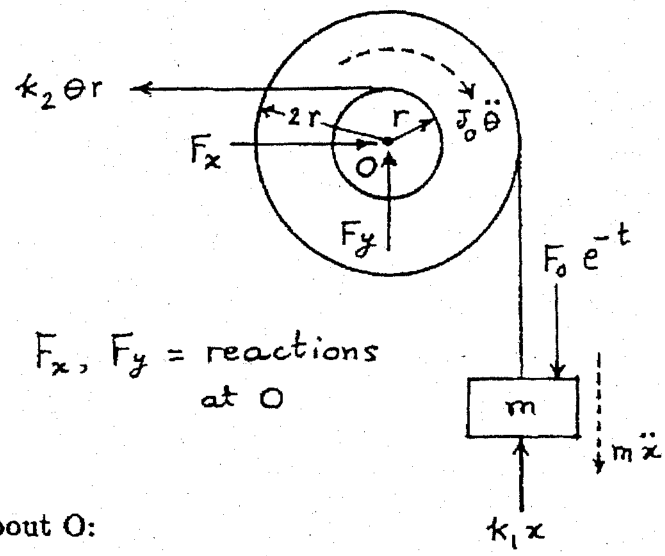
$$\theta(t) = \frac{1}{(1.4583)\,(46.2915)} \int_0^t (37.5\, e^{-\tau}) \sin 46.2915\,(t - \tau)\, d\tau \tag{3}$$

Using Eq. (5) in the solution of Problem 4.31, Eq. (3) can be expressed as

$$\theta(t) = -\,0.5555\, e^{-t} \left[\frac{e^{(t-\tau)}}{1^2 + 46.2915^2} \left\{ \sin 46.2915\,(t-\tau) - 46.2915 \cos\,(t-\tau) \right\} \right]_{\tau=0}^{t}$$

$$= 0.01199\, e^{-t} + 2.591\,(10^{-4}) \sin 46.2915\, t - 0.01199 \cos 46.2915\, t \text{ radian} \tag{4}$$

4.34



Equation of motion for rotation of pulley about O:

$$J_0\, \ddot{\theta} + m\, \ddot{x}\, (2\, r) + k_1\, x\, (2\, r) + k_2\, (\theta\, r)\, r = 2\, r\, F_0\, e^{-t} \tag{1}$$

where $\theta = \frac{x}{2\,r}$. Eq. (1) can be rewritten in terms of x only as:

$$\left(\frac{J_0}{2\,r} + 2\, m\, r \right) \ddot{x} + \left(2\, k_1\, r + \frac{1}{2}\, k_2\, r \right) x = 2\, r\, F_0\, e^{-t} \tag{2}$$

For given data, Eq. (2) becomes

$$11.0\, \ddot{x} + 112.5\, x = 5\, e^{-t} \tag{3}$$

Noting that the system is undamped with $\omega_n = \sqrt{\frac{112.5}{11.0}} = 3.1980$ rad/sec, the convolution integral, Eq. (4.33), can be used to find the steady state response as:

$$x(t) = -\,0.1421\, e^{-t} \int_0^t e^{(t-\tau)} \sin 3.1980\,(t - \tau))\,(-\,d\tau) \tag{4}$$

Using Eq. (5) in the solution of Problem 4.31, Eq. (4) can be expressed as

$$x(t) = -\frac{0.1421\ e^{-t}}{1^2 + 3.1980^2}\left[e^{(t-\tau)} \left\{ \sin 3.1980\,(t-\tau) - 3.1980 \cos 3.1980\,(t-\tau) \right\} \right]_{\tau=0}^{t}$$

$$= 0.04048\ e^{-t} + 0.01266 \sin 3.1980\, t - 0.04048 \cos 3.1980\, t \text{ m} \tag{5}$$

4.35

$$F(t) = \begin{cases} F_0 & ; \quad 0 \le t \le t_0 \\ 0 & ; \quad t > t_0 \end{cases} \tag{E$_1$}$$

Eq. (4.33) gives, for an undamped system,

$$x(t) = \frac{1}{m\,\omega_n} \int_0^t F(\tau) \sin \omega_n (t-\tau)\, d\tau \tag{E$_2$}$$

For $0 \le t \le t_0$:

$$x(t) = \frac{F_0}{m\,\omega_n} \int_0^t \sin \omega_n (t-\tau)\, d\tau = \frac{F_0}{k}\left(1 - \cos \omega_n t\right) \tag{E$_3$}$$

using Eq. (E_5) in the solution of problem 4.26.

For $t > t_0$:

$$x(t) = \frac{F_0}{m\,\omega_n} \int_0^{t_0} \sin \omega_n (t-\tau)\, d\tau$$

Using the relation

$$\int_0^{t_0} \sin \omega_n (t-\tau) \cdot d\tau = \frac{1}{\omega_n}\left\{ \cos \omega_n (t-t_0) - \cos \omega_n t \right\}$$

the solution can be expressed as

$$x(t) = \frac{F_0}{k}\left[\cos \omega_n (t-t_0) - \cos \omega_n t \right] \tag{E$_4$}$$

Response spectrum:

For $0 \le t \le t_0$,

$$x(t) = \frac{F_0}{k}\left(1 - \cos \omega_n t\right) \tag{E$_5$}$$

$$\frac{dx}{dt} = \frac{F_0\,\omega_n}{k} \sin \omega_n t = 0 \Rightarrow \omega_n t_{max} = \pi$$

$$\therefore x_{max} = x(t = t_{max}) = \frac{F_0}{k}\left(1 - \cos \omega_n t_{max}\right) = \frac{2 F_0}{k} \tag{E$_6$}$$

For $t > t_0$,

$$x(t) = \frac{F_0}{k}\left\{ \cos \omega_n (t-t_0) - \cos \omega_n t \right\} \tag{E$_7$}$$

$$\frac{dx}{dt} = -\frac{F_0\,\omega_n}{k}\left\{ \sin \omega_n (t-t_0) - \sin \omega_n t \right\} = 0$$

$$\Rightarrow \quad \sin \omega_n (t_{max} - t_o) = \sin \omega_n t_{max}$$

i.e.,
$$\tan \omega_n t_{max} = \left(\frac{\sin \omega_n t_o}{\cos \omega_n t_o - 1} \right) \tag{E8}$$

i.e.,
$$\sin \omega_n t_{max} = \frac{\sin \omega_n t_o}{\sqrt{2(1 - \cos \omega_n t_o)}} \tag{E9}$$

and
$$\cos \omega_n t_{max} = \frac{\cos \omega_n t_o - 1}{\sqrt{2(1 - \cos \omega_n t_o)}} \tag{E10}$$

$$\therefore \quad x_{max} = x(t = t_{max}) = \frac{F_o}{k} \{ \cos \omega_n t_{max} \cdot \cos \omega_n t_o + \sin \omega_n t_{max} \cdot \sin \omega_n t_o - \cos \omega_n t_{max} \}$$

$$= \frac{F_o}{k} \left\{ \frac{(\cos \omega_n t_o - 1)^2}{\sqrt{2(1 - \cos \omega_n t_o)}} + \frac{\sin^2 \omega_n t_o}{\sqrt{2(1 - \cos \omega_n t_o)}} \right\}$$

$$= \frac{2 F_o}{k} \sin \frac{\omega_n t_o}{2} \tag{E11}$$

Plotting: $\quad \omega_n = \dfrac{2\pi}{\tau_n}$

For $0 \le t \le t_o$, $\quad \omega_n t_{max} = \pi \quad$ or $\quad \dfrac{t_{max}}{\tau_n} = \dfrac{1}{2}$

when $t \le t_o$, $\quad t_{max} = \dfrac{\tau_n}{2} \le t_o$

i.e., $\quad \dfrac{t_o}{\tau_n} \ge \dfrac{1}{2}$

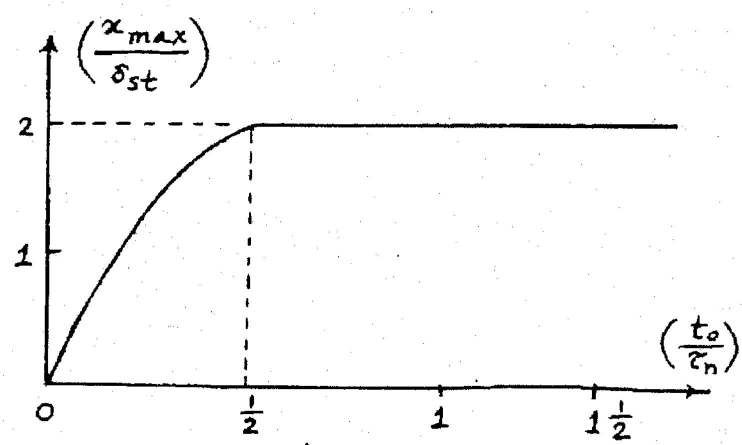
$$\therefore \quad \frac{x_{max}}{\delta_{st}} = 2 \quad \text{for} \quad \frac{t_o}{\tau_n} \ge \frac{1}{2} \tag{E12}$$

For $t > t_o$, $\quad \dfrac{t_o}{\tau_n} < \dfrac{1}{2}$

$$\frac{x_{max}}{\delta_{st}} = 2 \sin \frac{\omega_n t_o}{2} = 2 \sin \frac{2\pi t_o}{2 \tau_n}$$

$$\therefore \quad \frac{x_{max}}{\delta_{st}} = 2 \sin \frac{\pi t_o}{\tau_n} \quad \text{for} \quad \frac{t_o}{\tau_n} < \frac{1}{2} \tag{E13}$$

Eqs. (E_{12}) and (E_{13}) are plotted in the figure.



Response spectrum for a rectangular pulse-type load

(4.36) The response is found in problem 4.21.

For $0 \le t \le t_o$:

$$x(t) = \frac{F_o}{k}\left(1 - \cos\omega_n t\right) + \frac{F_o}{2\,m\,\omega_n}\cdot\frac{2\,\omega_n}{\left\{\left(\frac{\pi}{2t_o}\right)^2 - \omega_n^2\right\}}\cdot\left\{\cos\frac{\pi t}{2t_o} - \cos\omega_n t\right\} \quad (E_1)$$

For x_{max}, $\frac{dx}{dt} = 0$

i.e., $$\frac{F_o\,\omega_n}{k}\sin\omega_n t + \frac{F_o}{m\left\{\left(\frac{\pi}{2t_o}\right)^2 - \omega_n^2\right\}}\left(-\frac{\pi}{2t_o}\sin\frac{\pi t}{2t_o} + \omega_n\sin\omega_n t\right) = 0$$

which can be reduced to the form

$$\sin\omega_n t_{max} = \left[\frac{\pi}{2\,\omega_n t_o\left\{m\left(\frac{\pi}{2t_o}\right)^2 - m\,\omega_n^2\right\} + k}\right]\sin\frac{\pi\,t_{max}}{2t_o} \quad (E_2)$$

Once t_{max} is known from (E_2), Eq. (E_1) can be used to find x_{max} as:

$$x_{max} = x\left(t = t_{max}\right) = \frac{F_o}{k}\left(1 - \cos\omega_n t_{max}\right) + \frac{F_o}{m\left\{\left(\frac{\pi}{2t_o}\right)^2 - \omega_n^2\right\}}\left(\cos\frac{\pi\,t_{max}}{2t_o} - \cos\omega_n t_{max}\right) \quad (E_3)$$

For $t > t_0$:

$$x(t) = \frac{F_0}{k} \cos \omega_n (t - t_0)$$

$$- \sin \omega_n (t-t_0) \left\{ \frac{F_0}{2m\,\omega_n \left(\frac{\pi}{2t_0} - \omega_n\right)} + \frac{F_0}{2m\,\omega_n\left(\frac{\pi}{2t_0} + \omega_n\right)} \right\}$$

$$+ \cos \omega_n t \left\{ - \frac{F_0}{k} - \frac{F_0}{2m\,\omega_n \left(\frac{\pi}{2t_0} - \omega_n\right)} + \frac{F_0}{2m\,\omega_n \left(\frac{\pi}{2t_0} + \omega_n\right)} \right\} \qquad (E_4)$$

For t_{max}, $\frac{dx}{dt} = 0$

i.e.,

$$- \frac{F_0\,\omega_n}{k} \sin \omega_n (t_{max} - t_0) - \frac{F_0\,\pi\,\omega_n \cos \omega_n (t_{max} - t_0)}{2m\,\omega_n\,t_0 \left\{ \left(\frac{\pi}{2t_0}\right)^2 - \omega_n^2 \right\}}$$

$$+ \frac{F_0 \left(\frac{\pi}{2t_0}\right)^2 \omega_n}{k \left\{ \left(\frac{\pi}{2t_0}\right)^2 - \omega_n^2 \right\}} \sin \omega_n t_{max} = 0 \qquad (E_5)$$

Once t_{max} is found by solving Eq. (E_5), x_{max} can be found from Eq. (E_4) as

$$x_{max} = x(t = t_{max}) = \frac{F_0}{k} \cos \omega_n (t_{max} - t_0)$$

$$- \frac{\pi F_0}{2\,\omega_n\,m\,t_0 \left\{ \left(\frac{\pi}{2t_0}\right)^2 - \omega_n^2 \right\}} \sin \omega_n (t_{max} - t_0)$$

$$- \frac{F_0 \left(\frac{\pi}{2t_0}\right)^2}{k \left\{ \left(\frac{\pi}{2t_0}\right)^2 - \omega_n^2 \right\}} \cos \omega_n t_{max} \qquad (E_6)$$

Eqs. (E_3) and (E_6) can be used to plot x_{max} versus ω_n to get the displacement response spectrum.

4.37 Base acceleration $= \ddot{y}(t) = a_0 \left(1 - \sin \frac{\pi t}{2 t_0}\right) \qquad (E_1)$

For an undamped system, the relative displacement is given by Eq. (4.36):

$$z(t) = - \frac{1}{\omega_n} \int_0^t \ddot{y}(\tau) \sin \omega_n (t - \tau)\, d\tau$$

$$= - \frac{1}{\omega_n} \left[\int_0^t a_0 \sin \omega_n (t - \tau)\, d\tau - \int_0^t a_0 \sin \frac{\pi t}{2 t_0} \sin \omega_n (t - \tau)\, d\tau \right] \qquad (E_2)$$

Here

$$\int_0^t \sin \omega_n (t-\tau)\, d\tau = \left(\frac{1-\cos \omega_n t}{\omega_n} \right) \qquad (E_3)$$

from Eq. (E_5) in the solution of problem 4.26.

and

$$\int_0^t \sin \frac{\pi \tau}{2 t_0} \left(\sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau \right) d\tau$$

$$= \sin \omega_n t \left\{ - \frac{\cos\left(\frac{\pi}{2t_0} - \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} - \frac{\cos\left(\frac{\pi}{2t_0} + \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} \right\}_0^t$$

$$- \cos \omega_n t \left\{ \frac{\sin\left(\frac{\pi}{2t_0} - \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} - \frac{\sin\left(\frac{\pi}{2t_0} + \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} \right\}_0^t \qquad (E_4)$$

Thus the solution, Eq. (E_2), can be finally expressed as

$$z(t) = -\frac{a_0}{\omega_n}\left(1 - \frac{\cos \omega_n t}{\omega_n}\right) - \frac{a_0}{\omega_n}\left\{ \sin \omega_n t \cdot \left[\frac{\cos\left(\frac{\pi}{2t_0} - \omega_n\right)t}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} + \frac{\cos\left(\frac{\pi}{2t_0} + \omega_n\right)t}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} - 2 \right] + \cos \omega_n t \cdot \left[\frac{\sin\left(\frac{\pi}{2t_0} - \omega_n\right)t}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} - \frac{\sin\left(\frac{\pi}{2t_0} + \omega_n\right)t}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} \right] \right\} \qquad (E_5)$$

4.38 During $0 \le t \le t_0$:

$$x(t) = \frac{F_0}{k}\left(1 - \frac{t}{t_0} - \cos \omega_n t + \frac{1}{\omega_n t_0} \sin \omega_n t\right) \qquad (E_1)$$

$\dot{x}(t) = 0$ gives $\omega_n t_0 \sin \omega_n t_m = 1 - \cos \omega_n t_m$

i.e. $\omega_n t_m = 2 \tan^{-1}(\omega_n t_0)$ $\qquad (E_2)$

(E_1) becomes

$$\frac{x_m}{(F_0/k)} = 1 - \frac{t_m}{t_0} - \cos \omega_n t_m + \frac{1}{\omega_n t_0} \sin \omega_n t_m \qquad (E_3)$$

where t_m is given by (E_2).

During $t > t_0$:

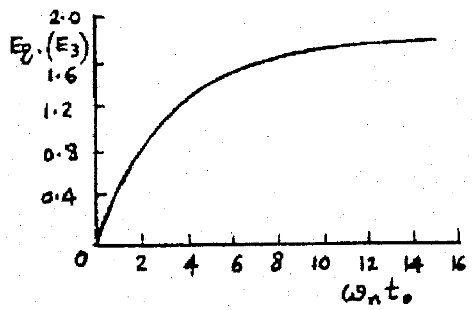
$$x(t) = \frac{F_0}{k \omega_n t_0}\left[(1 - \cos \omega_n t_0) \sin \omega_n t - (\omega_n t_0 - \sin \omega_n t_0) \cos \omega_n t\right] \qquad (E_4)$$

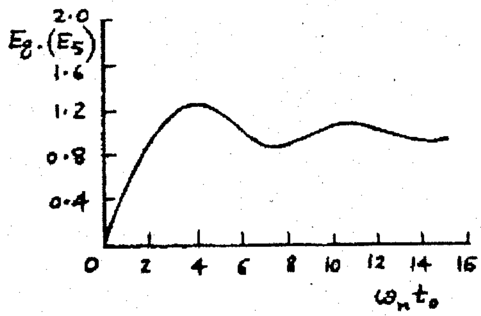
i.e. $\dfrac{x(t)\, k \omega_n t_0}{F_0} = \underset{\sim}{x}(t) = A \sin \omega_n t + B \cos \omega_n t$

where $A = 1 - \cos \omega_n t_0$; $B = -(\omega_n t_0 - \sin \omega_n t_0)$

Since $\underset{\sim}{x}\big|_{max} = \sqrt{A^2 + B^2}$,

$$\frac{x_m}{(F_0/k)} = \frac{1}{\omega_n t_0}\left[(1 - \cos \omega_n t_0)^2 + (\omega_n t_0 - \sin \omega_n t_0)^2\right]^{1/2} \qquad (E_5)$$





4.39 From Example 4.7, the response of the building frame is given by

$$x(t) = \frac{F_o}{k}\left[1 - \frac{t}{t_o} - \cos\omega_n t + \frac{1}{\omega_n t_o}\sin\omega_n t\right], \quad 0 \le t \le t_o \qquad (E_1)$$

and

$$x(t) = \frac{F_o}{k\,\omega_n t_o}\left[(1-\cos\omega_n t_o)\sin\omega_n t - (\omega_n t_o - \sin\omega_n t_o)\cos\omega_n t\right], \quad t > t_o \qquad (E_2)$$

(i) For $0 \le t \le t_o$:

For $x(t)$ to be maximum, the quantity inside square brackets in (E_1) must be maximum. This implies that

$$\frac{d}{dt}\left[1 - \frac{t}{t_o} - \cos\omega_n t + \frac{1}{\omega_n t_o}\sin\omega_n t\right] = 0$$

i.e., $\omega_n t_o \sin\omega_n t = 1 - \cos\omega_n t$

i.e., $\omega_n t_o \cos\frac{\omega_n t}{2} = \sin\frac{\omega_n t}{2}$

i.e., $$\tan\frac{\omega_n t}{2} = \omega_n t_o \qquad (E_3)$$

Thus, if $x(t)$ attains its maximum value at $t = t_{max}$, Eq.(E_3) gives

$$t_{max} = \frac{2}{\omega_n}\tan^{-1}(\omega_n t_o) \qquad (E_4)$$

Once t_{max} is known from (E_4), (E_1) gives

$$x_{max} = x(t = t_{max}) = \frac{F_o}{k}\left\{1 - \frac{t_{max}}{t_o} - \cos\omega_n t_{max} + \frac{1}{\omega_n t_o}\sin\omega_n t_{max}\right\} \qquad (E_5)$$

(ii) For $t > t_o$:

For $x(t)$ to be maximum, the quantity inside square brackets in Eq. (E_2) must be maximum. This implies that

$$\frac{d}{dt}\left[(1-\cos\omega_n t_o)\sin\omega_n t - (\omega_n t_o - \sin\omega_n t_o)\cos\omega_n t\right] = 0$$

i.e.,

$$(1-\cos\omega_n t_0)\,\omega_n\,\cos\omega_n t + (\omega_n t_0 - \sin\omega_n t_0)\,\omega_n \sin\omega_n t = 0$$

i.e.,

$$\tan\omega_n t = -\left(\frac{1-\cos\omega_n t_0}{\omega_n t_0 - \sin\omega_n t_0}\right) \tag{E_6}$$

If $x(t)$ attains its maximum at $t = t_{max}$, Eq. (E_6) gives

$$t_{max} = \frac{1}{\omega_n}\tan^{-1}\left(\frac{-1+\cos\omega_n t_0}{\omega_n t_0 - \sin\omega_n t_0}\right) \tag{E_7}$$

Once t_{max} is computed from (E_7), (E_2) gives

$$x_{max} = x(t=t_{max}) = \frac{F_0}{k\,\omega_n t_0}\left[(1-\cos\omega_n t_0)^2 + (\omega_n t_0 - \sin\omega_n t_0)^2\right]^{\frac{1}{2}} \tag{E_8}$$

Given data:

$m = 5000$ kg, $F_0 = 4\times10^6$ N, $t_0 = 0.4$ sec, $x_{max} \le 0.01$ m.

$\omega_n = \sqrt{k/m} = 0.01414\sqrt{k}$

Procedure:

1. Assume a series of values of k.
2. Find t_{max} using Eqs. (E_4) and (E_7).
3. Find x_{max} using Eqs. (E_5) and (E_8).
4. Select k such that $x_{max} \le 0.01$ m in Eq. (E_5) or (E_8).

Sample computer program and results are shown below.

```
      XM=5000.0
      FO=4.0E+6
      TO=0.4
      XK=0.0
      DO 10 I=1,100
      XK=XK+1.0E+7
      OMN=0.01414*SQRT(XK)
      TMAX1=(2.0/OMN)*ATAN(OMN*TO)
      XMAX1=(FO/XK)*(1.0-(TMAX1/TO)-COS(OMN*TMAX1)+SIN(OMN*TMAX1)/
     2 (OMN*TO))
      XNR=-(1.0-COS(OMN*TO))
      XDR=(OMN*TO-SIN(OMN*TO))
      TMAX2=ATAN(XNR/XDR)/OMN
      X1=(1.0-COS(OMN*TO))**2
      X2=(OMN*TO-SIN(OMN*TO))**2
      X3=(X1+X2)**0.5
      XMAX2=X3*FO/(XK*OMN*TO)
      PRINT 5,I,XK,TMAX1,XMAX1,TMAX2,XMAX2
5     FORMAT(I5,2X,E15.4,2X,2E12.4,2X,2E12.4)
10    CONTINUE
      STOP
      END
```

1	0.1000E+08	0.6776E-01	0.7322E+00	-0.5134E-03	0.41[illegible]E+00
2	0.2000E+08	0.4843E-01	0.3758E+00	-0.8204E-05	0.19[illegible]E+00
3	0.3000E+08	0.3973E-01	0.2534E+00	-0.3859E-04	0.13[illegible]E+00
4	0.4000E+08	0.3450E-01	0.1914E+00	-0.4108E-03	0.10[illegible]E+00
5	0.5000E+08	0.3092E-01	0.1538E+00	-0.4234E-03	0.78[illegible]E-01
⋮					

These results indicate that the required stiffness is

$k = 8 \times 10^8$ N/m.

4.40 Let d = thickness of bracket. Then, from Example 4.12, self weight of beam = w = 0.5 d lb, total weight at free end of beam = W = 0.5 d + 0.4 lb, moment of inertia of beam cross section = I = 0.04167 d^3 in^4, static deflection of beam under W:

$$\delta_{st} = \frac{W\,\ell^3}{3\,E\,I} = \frac{(0.5\,d + 0.4)}{d^3}\ 7.9994\ (10^{-4})\ \text{in}$$

We need to use a trial and error procedure to find the correct value of d.

Let d = 1 in:

$w = 0.5$ lb, $W = 0.9$ lb, $I = 0.04167\ in^4$, $\delta_{st} = 0.9\ (7.9994)\ (10^{-4}) = 7.19946\ (10^{-4})$ in,

$$\tau_n = 2\,\pi \sqrt{\frac{\delta_{st}}{g}} = \sqrt{\frac{7.19946\ (10^{-4})}{386.4}} = 0.008577 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.008577} = 11.6591$$

From Fig. 4.49, shock amplification factor (A_a) corresponding to $t_0/\tau_n = 11.6591$ is $A_a \approx 2.0$.

Dynamic load at end of cantilever $= P_d = A_a\ M\ a_s = (2.0)\ (0.9/g)\ (100g) = 180.0$ lb.

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{(180\ (10))\ (1.0/2)}{0.04167} = 21598.2721\ \text{lb/in}^2$$

Since this is smaller than the permissible value of 26000 psi, we choose a smaller value of d next.

Let d = 0.9 in:

$$w = 0.45 \text{ lb},\ W = 0.85 \text{ lb},\ I = 0.03038\ \text{in}^4,\ \ \delta_{st} = \frac{0.85\ (7.9994\ (10^{-4}))}{0.9^3} = 9.3271\ (10^{-4})\ \text{in}$$

$$\tau_n = 2\,\pi \sqrt{\frac{\delta_{st}}{g}} = 2\,\pi \sqrt{\frac{9.3271\ (10^{-4})}{386.4}} = 0.009762 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.009762} = 10.2438$$

$A_a \approx 2.0$, $P_d = A_a\ M\ a_s = (2.0)\ (0.85/g)\ (100g) = 170.0$ lb

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{(170\ (10))\ (0.9/2)}{0.03038} = 25181.0402\ \text{lb/in}^2$$

Since this stress is close to the maximum permissible value, we take d = 0.9 in.

4.41 Let d = thickness of bracket. Then from Example 4.12, self weight of beam = w = 0.5 d lb, total weight at free end of beam = W = 0.5 d + 0.4 lb, moment of inertia of beam cross section = I = 0.04167 d^3 in^4, static deflection of beam under W:

$$\delta_{st} = \left(\frac{0.5\ d + 0.4}{d^3} \right) 7.9994\ (10^{-4})\ in$$

We need to use a trial and error procedure to find the correct value of d. However, since the shock amplification factor, for large values of t_0/τ_n, for the triangular pulse of Fig. 4.50 is similar to that of the pulse shown in Fig. 4.11, we start with d = 0.6 in. This gives:

w = 0.3 lb, W = 0.7 lb, I = (0.04167) (0.216) = 0.009001 in^4, $\delta_{st} = \frac{0.7}{0.6^3} (7.9994\ (10^{-4})) = 25.9240\ (10^{-4})$ in, $\tau_n = 2\pi \sqrt{\frac{25.9240\ (10^{-4})}{386.4}} =$ 0.01627 sec, $t_0/\tau_n = (0.1/0.01627) = 6.1445$. From Fig. 4.50, we find shock amplification factor as $A_a \approx 1.1$, dynamic load at end of beam = $P_d = A_a\ M\ a_a = (1.1)\ (0.7/g)\ (100g) = 77.0$ lb, maximum bending stress at root of beam = $\sigma_{max} = \frac{M_b\ c}{I} = (77(10))\ (0.6/2)\ /\ (0.009001) = 25663.8151$ psi. Since this stress is very close to the maximum permissible value, we select d = 0.6 in as the design.

4.42 Let d = thickness of bracket (beam). Self weight of beam = w = d (1/2) (16) (0.1) = 0.8 d lb, total load at middle of beam = W = (1 + 0.8 d) lb, area moment of inertia of beam cross section = I = $\frac{1}{12} (\frac{1}{2})\ d^3\ in^4$. Static deflection of beam at middle due to W:

$$\delta_{st} = \frac{W\ \ell^3}{192\ E\ I} = \frac{(1 + 0.8\ d)\ (16^3)}{192\ (10^7)\ (0.04167\ d^3)} = (1 + 0.8\ d)\ (51.1959\ (10^{-6}))\ in$$

We need to use a trial and error procedure to determine the correct value of d.

Let d = 0.4 in:

w = 0.32 lb, W = 1.32 lb, I = 0.002667 in^4, $\delta_{st} = 67.5786\ (10^{-6})$ in,

$$\tau_n = 2\pi \sqrt{\frac{67.5786\ (10^{-6})}{386.4}} = 0.002628\ sec$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.002628} = 38.0569$$

From Fig. 4.11(b), $A_a \approx 1.1$, and the dynamic load on beam is given by $P_d = A_a\ M\ a_s$ where M = total mass of beam and a_s = acceleration due to shock = 100 g. Thus $P_d = (1.1)\ (1.32/g)\ (100\ g) = 145.2$ lb. Maximum bending moment in a fixed-fixed beam due to load (F) at the middle is given by $M_b = \frac{F\ \ell}{8}$ so that

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{\left(\frac{145.2\ (16)}{8} \right) (\frac{0.4}{2})}{0.002667} = 21777.2778\ lb/in^2$$

Since this stress is smaller than the maximum permissible value of 26000 psi, we next select a smaller value of d.

Let d = 0.35 in:

$w = 0.28$ lb, $W = 1.28$ lb, $I = 0.001787$ in^4, $\delta_{st} = 65.5307\ (10^{-6})$ in,

$$\tau_n = 2\,\pi \sqrt{\frac{65.5307\ (10^{-6})}{386.4}} = 0.002587 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.002587} = 38.6469$$

From Fig. 4.11(b), $A_a \approx 1.1$, $P_d = (1.1)\ (1.28/g)\ (100g) = 140.8$ lb

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{\left(\frac{140.8\ (16)}{8}\right)\left(\frac{0.35}{2}\right)}{0.001787} = 27576.9446 \text{ lb/in}^2$$

Since this stress exceeds the permissible value, we increase the value of d.

Let d = 0.37 in:

$w = 0.296$ lb, $W = 1.296$ lb, $I = 0.002111$ in^4, $\delta_{st} = 66.3499\ (10^{-6})$ in

$$\tau_n = 2\,\pi \sqrt{\frac{66.3499\ (10^{-6})}{386.4}} = 0.002604 \text{ sec}$$

$$\frac{t_0}{\tau_n} = 38.4024$$

From Fig. 4.11(b), $A_a \approx 1.1$, $P_d = (1.1)\ (1.296/g)\ (100\ g) = 142.56$ lb, and

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{\left(\frac{142.56\ (16)}{8}\right)\left(\frac{0.37}{2}\right)}{0.002111} = 24986.8309 \text{ lb/in}^2$$

Since this stress is close to the maximum permissible value, we select the design as d = 0.37 in.

4.43 $W = m\ g = 100000$ lb, $\varsigma = 0.05$, $\sigma_y = 30000$ psi, σ_{max} = maximum permissible stress $= \frac{\sigma_y}{2} = \frac{30000}{2} = 15000$ psi,

$$\tau_n = \frac{2\,\pi}{\omega_n} = 2\,\pi\sqrt{\frac{m}{k}} = 2\,\pi\sqrt{\frac{100000}{386.4\ k}} = \frac{101.0793}{\sqrt{k}} \text{ sec}$$

We need to use a trial and error procedure to find k.

Let k = 10000 lb/in:

$$k = 10^4 = \frac{3\,E\,I}{\ell^3} = \frac{3\,(30\,(10^6))\,I}{600^3}$$

$$I = 24000 \text{ in}^4 = \frac{\pi}{64}\, d_o^4\, (0.5904) \text{ with } \frac{d_i}{d_o} = 0.8$$

$$d_o^4 = 82.8121\,(10^4) \text{ in}^4$$

$$d_o = 30.1664 \text{ in} \; ; \; d_i = 24.1331 \text{ in}$$

$$\tau_n = \frac{101.0793}{\sqrt{10^4}} \approx 1 \text{ sec}$$

From Fig. 4.14, for $\tau_n = 1$ sec and $\varsigma = 0.05$, we find $S_v = 25$ in/sec, $S_d = 4.2$ in and $S_a = 0.42$ g.

Maximum shear force in column:

$$F_{max} = \frac{W}{g}\, S_a = \frac{100000}{g}\,(0.42g) = 42000 \text{ lb}$$

Maximum bending moment:

$$M_b = F_{max}\, h = (42000)\,(50\,(12)) = 25.2\,(10^6) \text{ lb--in}$$

Maximum bending stress:

$$\sigma_b = \frac{M_b\, c}{I} = \frac{(25.2\,(10^6))\,(30.1664/2)}{24\,(10^3)} = 15837.36 \text{ lb/in}^2$$

Since this stress is slightly smaller than the maximum permissible value, we choose a larger value of k.

Let k = 20000 lb/in:

$$k = 2\,(10^4) = \frac{3\,(30\,(10^6))\,I}{600^3}$$

$$I = 48000 \text{ in}^4 = \frac{\pi}{64}\, d_o^4\, (0.5904)$$

$$d_o^4 = 165.6243\,(10^4) \text{ in}^4$$

$$d_o = 35.8741 \text{ in} \; ; \; d_i = 28.6993 \text{ in}$$

$$\tau_n = \frac{101.0793}{\sqrt{20000}} = 0.7147 \text{ sec}$$

From Fig. 4.14, we find

$$S_v = 26 \text{ in/sec}, \; S_d = 3 \text{ in}, \; S_a = 0.6 \text{ g}$$

Maximum shear force in column:

$$F_{max} = \frac{W}{g}\, S_a = \frac{10^5}{g}\,(0.6g) = 60000 \text{ lb}$$

Maximum bending moment:

$$M_b = F_{max}\, h = (60000)\,(600) = 36\,(10^6) \text{ lb--in}$$

Maximum bending stress:

$$\sigma_b = \frac{M_b\, c}{I} = \frac{(36\,(10^6))\,(35.8741/2)}{48\,(10^3)} = 13452.7875 \text{ lb/in}^2$$

Since this stress is less than the maximum permissible value, we choose the inner and outer diameters of the column as $d_i = 28.6993$ in and $d_o = 35.8741$ in.

(4.44) m g = 5000 lb, $\varsigma = 0.02$. From Fig. 4.15, in order to have $S_a \approx 1$ g, we need to have $\tau_n = 0.2$ sec. Thus

$$\tau_n = 0.2 = \frac{2\pi}{\omega_n} \quad ; \quad \omega_n = 31.416 \text{ rad/sec} = \sqrt{\frac{k}{m}}$$

$$k = (31.416)^2 \; m = (31.416^2)\,(5000/386.4) = 12771.2870 \text{ lb/in}$$

(4.45) Equation of motion is $\ddot{x} + \omega_n^2 x = \frac{F_0}{m} e^{i\omega t}$

For zero initial conditions

$$(s^2 + \omega_n^2)\,\bar{x}(s) = \frac{F_0}{m} \cdot \frac{1}{s - i\omega} \quad ; \quad \bar{x}(s) = \frac{F_0}{m} \cdot \frac{1}{(s^2 + \omega_n^2)} \cdot \frac{s + i\omega}{(s^2 + \omega^2)}$$

Inverse Laplace transformation gives

$$x(t) = \frac{F_0}{m} \frac{1}{(\omega_n^2 - \omega^2)} \left\{ e^{i\omega t} - \left(\cos\omega_n t + \frac{i\omega}{\omega_n} \sin\omega_n t \right) \right\}$$

The terms containing $\cos\omega_n t$ and $\sin\omega_n t$ denote the transient part of the response and hence can be neglected. Thus the steady state response can be expressed as

$$x(t) = \frac{F_0}{k} \left(\frac{1}{1 - r^2} \right) e^{i\omega t} \qquad \text{where} \quad r = \omega/\omega_n .$$

(4.46) $\bar{F}(s) = \frac{F_0}{s}$

From Eq. (4.57),

$$\bar{x}(s) = \frac{F_0}{m s\,(s^2 + 2\zeta\omega_n s + \omega_n^2)} + \left(\frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) x_0 + \left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) \dot{x}_0$$

Inverse transformation gives

$$x(t) = \frac{F_0}{m} \left\{ 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi_1) \right\} + \frac{x_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1) + \frac{\dot{x}_0}{\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t$$

where $\phi_1 = \cos^{-1}(\zeta)$.

(4.47) The forcing function can be expressed as

$$F(t) = F_0 \left\{ u(t) - u(t - t_0) \right\}$$

where $u(t - \tau)$ is the unit step function applied at $t = \tau$. The Laplace transform of $F(t)$ is

$$\bar{F}(s) = F_0 \left(\frac{1}{s} - \frac{1}{s} e^{-s t_0} \right)$$

The equation of motion $m\ddot{x} + kx = F(t)$ gives

$$\bar{x}(s) = \frac{F_0}{s}\left(1 - e^{-st_0}\right)\frac{1}{ms^2 + k} = \frac{F_0}{m}\left\{\frac{1}{s(s^2+\omega_n^2)} - \frac{e^{-st_0}}{s(s^2+\omega_n^2)}\right\}$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \frac{1}{a^2}(1 - \cos at)$,

$$x(t) = \frac{F_0}{m\omega_n^2}(1 - \cos\omega_n t)\cdot u(t) - \frac{F_0}{m\omega_n^2}\{1 - \cos\omega_n(t-t_0)\}\, u(t-t_0)$$

Hence

$$x(t) = \begin{cases} \dfrac{F_0}{m\omega_n^2}(1 - \cos\omega_n t) & \text{for } 0 \le t \le t_0 \\ \dfrac{F_0}{m\omega_n^2}\{\cos\omega_n(t-t_0) - \cos\omega_n t\} & \text{for } t \ge t_0 \end{cases}$$

4.48

$$\dot{x}(t) = \frac{1}{k}\sum_{i=1}^{j-1}\Delta F_i\Big\{-(-\zeta\omega_n)e^{-\zeta\omega_n(t-t_i)}\cos\omega_d(t-t_i)$$
$$+ e^{-\zeta\omega_n(t-t_i)}\omega_d\sin\omega_d(t-t_i) - (-\zeta\omega_n)e^{-\zeta\omega_n(t-t_i)}\frac{\zeta\omega_n}{\omega_d}\sin\omega_d(t-t_i)$$
$$- e^{-\zeta\omega_n(t-t_i)}\frac{\zeta\omega_n}{\omega_d}\cdot\omega_d\cdot\cos\omega_d(t-t_i)\Big\}$$
$$= \frac{1}{k}\sum_{i=1}^{j-1}\Delta F_i\left\{e^{-\zeta\omega_n(t-t_i)}\right\}\left(\omega_d + \frac{\zeta^2\omega_n^2}{\omega_d}\right)\sin\omega_d(t-t_i)$$

$$\dot{x}_j = \dot{x}(t=t_j) = \frac{1}{k}\sum_{i=1}^{j-1}\Delta F_i\left(\omega_d + \frac{\zeta^2\omega_n^2}{\omega_d}\right)e^{-\zeta\omega_n(t_j-t_i)}\sin\omega_d(t_j-t_i)$$

4.49 Eq. (4.68): Differentiating Eq. (4.66),

$$\dot{x}(t) = \frac{F_j}{k}e^{-\zeta\omega_n(t-t_{j-1})}\omega_d\left(1 + \frac{\zeta^2\omega_n^2}{\omega_d^2}\right)\sin\omega_d(t-t_{j-1}) + \omega_d\, e^{-\zeta\omega_n(t-t_{j-1})} *$$
$$\Big\{-x_{j-1}\sin\omega_d(t-t_{j-1}) + \frac{\dot{x}_{j-1}}{\omega_d}\cos\omega_d(t-t_{j-1})$$
$$- \frac{\zeta\omega_n}{\omega_d^2}(\dot{x}_{j-1} + \zeta\omega_n x_{j-1})\sin\omega_d(t-t_{j-1})\Big\}$$

For $t = t_j$,

$$\dot{x}_j = \frac{F_j}{k}e^{-\zeta\omega_n\Delta t_j}\omega_d\left(1 + \frac{\zeta^2\omega_n^2}{\omega_d^2}\right)\sin\omega_d\Delta t_j + \omega_d\, e^{-\zeta\omega_n\Delta t_j}\Big\{-x_{j-1}\sin\omega_d\Delta t_j$$
$$+ \frac{\dot{x}_{j-1}}{\omega_d}\cos\omega_d\Delta t_j - \frac{\zeta\omega_n}{\omega_d^2}(\dot{x}_{j-1} + \zeta\omega_n x_{j-1})\sin\omega_d\Delta t_j\Big\}$$

Eq. (4.71):

$$\dot{x}(t) = \frac{\Delta F_j}{k\,\Delta t_j}\Big[1 - \zeta\omega_n e^{-\zeta\omega_n(t-t_{j-1})}\frac{2\zeta}{\omega_n}\cos\omega_d(t-t_{j-1}) - e^{-\zeta\omega_n(t-t_{j-1})} *$$
$$\frac{2\zeta}{\omega_n}\cdot\omega_d\cdot\sin\omega_d(t-t_{j-1}) + \zeta\omega_n e^{-\zeta\omega_n(t-t_{j-1})}\left(\frac{\omega_d^2 - \zeta^2\omega_n^2}{\omega_n^2\omega_d}\right)\sin\omega_d(t-t_{j-1})$$

$$- e^{-\zeta\omega_n(t-t_{j-1})}\left(\frac{\omega_d^2-\zeta^2\omega_n^2}{\omega_n^2\,\omega_d}\right)\omega_d\ \cos\omega_d(t-t_{j-1})\Big] + \frac{F_{j-1}}{k}\Big[\zeta\omega_n\, e^{-\zeta\omega_n(t-t_{j-1})} *$$

$$\cos\omega_d(t-t_{j-1}) + e^{-\zeta\omega_n(t-t_{j-1})}\,\omega_d\ \sin\omega_d(t-t_{j-1}) + \zeta\omega_n\, e^{-\zeta\omega_n(t-t_{j-1})} *$$

$$\left(\frac{\zeta\omega_n}{\omega_d}\right)\cdot \sin\omega_d(t-t_{j-1}) - e^{-\zeta\omega_n(t-t_{j-1})}\left(\frac{\zeta\omega_n}{\omega_d}\right)\omega_d\cdot\cos\omega_d(t-t_{j-1})\Big]$$

$$- \zeta\omega_n\ e^{-\zeta\omega_n(t-t_{j-1})}\ x_{j-1}\ \cos\omega_d(t-t_{j-1}) - e^{-\zeta\omega_n(t-t_{j-1})}\ x_{j-1}\,\omega_d *$$

$$\sin\omega_d(t-t_{j-1}) - \zeta\omega_n\, e^{-\zeta\omega_n(t-t_{j-1})}\left(\frac{\dot{x}_{j-1}+\zeta\omega_n x_{j-1}}{\omega_d}\right)\sin\omega_d(t-t_{j-1})$$

$$+ e^{-\zeta\omega_n(t-t_{j-1})}\left(\frac{\dot{x}_{j-1}+\zeta\omega_n x_{j-1}}{\omega_d}\right)\omega_d\ \cos\omega_d(t-t_{j-1})$$

Simplification of this equation gives at $t = t_j$,

$$\dot{x}_j = \frac{\Delta F_j}{k\,\Delta t_j}\left[1 - e^{-\zeta\omega_n\Delta t_j}\left\{\cos\omega_d\,\Delta t_j + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d\,\Delta t_j\right\}\right]$$

$$+ \frac{F_{j-1}}{k}\ e^{-\zeta\omega_n\Delta t_j}\ \frac{\omega_n^2}{\omega_d}\ \sin\omega_d\,\Delta t_j$$

$$+ e^{-\zeta\omega_n\Delta t_j}\left\{\dot{x}_{j-1}\cos\omega_d\,\Delta t_j - \frac{\zeta\omega_n}{\omega_d}\left(\dot{x}_{j-1} + \frac{\omega_n}{\zeta}x_{j-1}\right)\sin\omega_d\,\Delta t_j\right\}$$

4.50

i	t_i	$\dot{x}_i$, Eq. (4.68)	$\dot{x}_i$, Eq. (4.71)
1	0	0	0
2	0.1π	0.276010E+00	0.275640E+00
3	0.2π	0.462605E+00	0.461687E+00
4	0.3π	0.549249E+00	0.547683E+00
5	0.4π	0.536630E+00	0.534405E+00
6	0.5π	0.435845E+00	0.433036E+00
7	0.6π	0.266613E+00	0.263378E+00
8	0.7π	0.547668E-01	0.513272E-01
9	0.8π	-0.170711E+00	-0.174093E+00
10	0.9π	-0.380784E+00	-0.383830E+00
11	π	-0.549289E+00	-0.551730E+00

4.51 Method 1:

$$x_j = \frac{1}{k}\sum_{i=1}^{j-1}\Delta F_i\left\{1-\cos\omega_n(t_j-t_i)\right\}$$

$$\dot{x}_j = \frac{1}{k}\sum_{i=1}^{j-1}\Delta F_i\ \omega_n\ \sin\omega_n(t_j-t_i)$$

Method 2:

$$x_j = \frac{F_j}{k}\left[1-\cos\omega_n\,\Delta t_j\right] + x_{j-1}\cos\omega_n\,\Delta t_j + \frac{\dot{x}_{j-1}}{\omega_n}\sin\omega_n\,\Delta t_j$$

$$\dot{x}_j = \frac{F_j\,\omega_n}{k}\sin\omega_n\,\Delta t_j + \omega_n\left\{-x_{j-1}\sin\omega_n\,\Delta t_j + \frac{\dot{x}_{j-1}}{\omega_n}\cos\omega_n\,\Delta t_j\right\}$$

Method 3:

$$x_j = \frac{\Delta F_j}{k\,\Delta t_j}\left\{\Delta t_j - \frac{1}{\omega_n}\sin\omega_n\,\Delta t_j\right\} - \frac{F_{j-1}}{k}\left(1-\cos\omega_n\,\Delta t_j\right) + x_{j-1}\cos\omega_n\,\Delta t_j$$

$$+ \frac{\dot{x}_{j-1}}{\omega_n}\ \sin\omega_n\,\Delta t_j$$

$$\dot{x}_j = \frac{\Delta F_j}{k\,\Delta t_j}\left(1-\cos\omega_n \Delta t_j\right) + \frac{F_{j-1}}{k}\,\omega_n \sin\omega_n \Delta t_j + \dot{x}_{j-1}\cos\omega_n \Delta t_j$$
$$- \omega_n x_{j-1} \sin \omega_n \Delta t_j$$

VALUE OF	METHOD #1 (FIG.4.18)	METHOD #1 (FIG.4.19)	METHOD #2 (FIG.4.20)	METHOD #3 (FIG.4.21)
I	X(I)	X(I)	X(I)	X(I)
2	0.489135E-01	0.412870E-01	0.451034E-01	0.463829E-01
3	0.183327E+00	0.153639E+00	0.168413E+00	0.170863E+00
4	0.374871E+00	0.311494E+00	0.342969E+00	0.346380E+00
5	0.590262E+00	0.485757E+00	0.537537E+00	0.541621E+00
6	0.794774E+00	0.646981E+00	0.720014E+00	0.724433E+00
7	0.955998E+00	0.768556E+00	0.860896E+00	0.865287E+00
8	0.104732E+01	0.829582E+00	0.936446E+00	0.940461E+00
9	0.105081E+01	0.817133E+00	0.931268E+00	0.934605E+00
10	0.959172E+00	0.727695E+00	0.840009E+00	0.842438E+00
11	0.776638E+00	0.566437E+00	0.668027E+00	0.669410E+00
I	XD(I)	XD(I)	XD(I)	XD(I)
2	0.309017E+00	0.260676E+00	0.284772E+00	0.284646E+00
3	0.539444E+00	0.448685E+00	0.493775E+00	0.493286E+00
4	0.669917E+00	0.547974E+00	0.608327E+00	0.607283E+00
5	0.690013E+00	0.552279E+00	0.620126E+00	0.618406E+00
6	0.601222E+00	0.465650E+00	0.531993E+00	0.529561E+00
7	0.416706E+00	0.301948E+00	0.357498E+00	0.354412E+00
8	0.159909E+00	0.833536E-01	0.119506E+00	0.115921E+00
9	-0.137878E+00	-0.161957E+00	-0.152198E+00	-0.156045E+00
10	-0.440725E+00	-0.402734E+00	-0.423989E+00	-0.427799E+00
11	-0.711751E+00	-0.615408E+00	-0.661862E+00	-0.665302E+00

4.52 $\omega_n = \sqrt{k/m} = \sqrt{50/2} = 5$ rad/s ; $\zeta = \frac{c}{2m\omega_n} = \frac{2}{2(2)(5)} = 0.1$

The problem-dependent data to be used in Program 5 and the output are given.

```
C FOLLOWING 11 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION F(11),FF(11),DELF(11),T(11),X(11),XD(11),X1(11),
     2 XD1(11),X2(11),XD2(11),X3(11),XD3(11),X4(11),XD4(11)
      DATA F/-8.,-12.,-15.,-13.,-11.,-7.,-4.,3.,10.,15.,18./
      DO 5 I=1,11
    5 FF(I)=F(I)
      XAI=0.1
      OMN=5.0
      DELT=0.1
      XK=50.0
C END OF PROBLEM-DEPENDENT DATA
```

VALUE OF	METHOD #1 (FIG.4.18)	METHOD #1 (FIG.4.19)	METHOD #2 (FIG.4.20)	METHOD #3 (FIG.4.21)
I	X(I)	X(I)	X(I)	X(I)

2	-0.947628E-02	-0.355361E-01	-0.284289E-01	-0.308375E-01
3	-0.415895E-01	-0.124570E+00	-0.110554E+00	-0.117557E+00
4	-0.888374E-01	-0.231948E+00	-0.224265E+00	-0.229133E+00
5	-0.129452E+00	-0.316847E+00	-0.330743E+00	-0.326279E+00
6	-0.140543E+00	-0.345356E+00	-0.391810E+00	-0.371424E+00
7	-0.106067E+00	-0.292742E+00	-0.381288E+00	-0.341045E+00
8	-0.149353E-01	-0.146366E+00	-0.284987E+00	-0.219841E+00
9	0.138845E+00	0.786267E-01	-0.100887E+00	-0.136816E-01
10	0.341364E+00	0.340894E+00	0.147643E+00	0.244535E+00
11	0.559359E+00	0.589762E+00	0.415128E+00	0.488339E+00

I	XD(I)	XD(I)	XD(I)	XD(I)
2	-0.363160E-01	-0.136185E+00	-0.547484E+00	-0.618556E+00
3	-0.879517E-01	-0.209522E+00	-0.105218E+01	-0.105547E+01
4	-0.960236E-01	-0.208906E+00	-0.117245E+01	-0.111170E+01
5	-0.627967E-01	-0.123358E+00	-0.916858E+00	-0.762976E+00
6	0.182155E-01	0.100267E-01	-0.289472E+00	-0.106971E+00
7	0.114511E+00	0.191935E+00	0.482526E+00	0.746090E+00
8	0.238516E+00	0.375367E+00	0.138800E+01	0.167033E+01
9	0.358701E+00	0.499280E+00	0.220328E+01	0.239624E+01
10	0.429269E+00	0.522310E+00	0.265575E+01	0.268039E+01
11	0.420343E+00	0.448691E+00	0.258326E+01	0.195527E+01

4.53 The problem-dependent data to be used in Program 5 and results are given.

```
C FOLLOWING 11 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION F(51),FF(51),DELF(51),T(51),X(51),XD(51),X1(51),
     2 XD1(51),X2(51),XD2(51),X3(51),XD3(51),X4(51),XD4(51)
      DATA F/0.,2.,4.,6.,8.,10.,12.,14.,16.,18.,20.,20.,20.,20.,
     2 20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,0.,0.,0.,0.,
     3 0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,
     4 0./
      DO 5 I=1,51
    5 FF(I)=F(I)
      XAI=0.0
      OMN=27.3861
      DELT=0.01
      XK=1500.0
      NP=51
C END OF PROBLEM-DEPENDENT DATA
```

VALUE OF	METHOD #1 (FIG.4.18)	METHOD #1 (FIG.4.19)	METHOD #2 (FIG.4.20)	METHOD #3 (FIG.4.21)
I	X(I)	X(I)	X(I)	X(I)
2	0.496882E-04	0.993764E-04	0.496882E-04	0.662924E-04
3	0.244738E-03	0.439787E-03	0.244738E-03	0.326397E-03
4	0.669987E-03	0.109524E-02	0.669987E-03	0.860303E-03
5	0.139312E-02	0.211625E-02	0.139312E-02	0.172759E-02
6	0.245961E-02	0.352610E-02	0.245961E-02	0.296301E-02
7	0.388935E-02	0.531910E-02	0.388935E-02	0.457384E-02
8	0.567516E-02	0.746097E-02	0.567516E-02	0.653940E-02
9	0.778331E-02	0.989115E-02	0.778330E-02	0.881258E-02
10	0.101560E-01	0.125288E-01	0.101560E-01	0.113233E-01

⋮

45	0.798152E-02	0.845167E-02	0.798155E-02	0.827565E-02
46	0.831680E-02	0.807062E-02	0.831682E-02	0.825346E-02
47	0.803221E-02	0.708804E-02	0.803222E-02	0.761612E-02
48	0.714897E-02	0.557718E-02	0.714896E-02	0.641114E-02
49	0.573290E-02	0.365064E-02	0.573288E-02	0.472831E-02
50	0.388954E-02	0.145202E-02	0.388951E-02	0.269308E-02
51	0.175629E-02	-0.854840E-03	0.175624E-02	0.457121E-03

I	XD(I)	XD(I)	XD(I)	XD(I)
2	0.360601E-03	0.721201E-03	0.987545E-02	0.148443E-01
3	0.105493E-02	0.174925E-02	0.288903E-01	0.385198E-01
4	0.203123E-02	0.300753E-02	0.556274E-01	0.692620E-01
5	0.321673E-02	0.440224E-02	0.880938E-01	0.104780E+00
6	0.452309E-02	0.582945E-02	0.123870E+00	0.142425E+00
7	0.585294E-02	0.718278E-02	0.160289E+00	0.179393E+00
8	0.710715E-02	0.836136E-02	0.194637E+00	0.212929E+00
9	0.819225E-02	0.927735E-02	0.224354E+00	0.240531E+00
10	0.902736E-02	0.986248E-02	0.247224E+00	0.260144E+00
⋮				
45	0.233952E-02	-0.244374E-03	0.640694E-01	0.289820E-01
46	0.937302E-04	-0.252103E-02	0.256574E-02	-0.333924E-01
47	-0.215904E-02	-0.460977E-02	-0.591291E-01	-0.932780E-01
48	-0.425090E-02	-0.635495E-02	-0.116417E+00	-0.146211E+00
49	-0.602593E-02	-0.762648E-02	-0.165028E+00	-0.188247E+00
50	-0.735183E-02	-0.832959E-02	-0.201339E+00	-0.216253E+00
51	-0.812978E-02	-0.841188E-02	-0.222644E+00	-0.228140E+00

(4.54) The problem-dependent data to be used in Program 5 and output are given below.

```
C FOLLOWING 11 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION F(51),FF(51),DELF(51),T(51),X(51),XD(51),X1(51),
     2 XD1(51),X2(51),XD2(51),X3(51),XD3(51),X4(51),XD4(51)
      DATA F/0.,2.,4.,6.,8.,10.,12.,14.,16.,18.,20.,20.,20.,20.,
     2 20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,0.,0.,0.,0.,
     3 0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,
     4 0./
      DO 5 I=1,51
    5 FF(I)=F(I)
      XAI=0.0912872
      OMN=27.3861
      DELT=0.01
      XK=1500.0
      NP=51
C END OF PROBLEM-DEPENDENT DATA
```

VALUE OF	METHOD #1 (FIG.4.18)	METHOD #1 (FIG.4.19)	METHOD #2 (FIG.4.20)	METHOD #3 (FIG.4.21)
I	X(I)	X(I)	X(I)	X(I)
2	0.488713E-04	0.977126E-04	0.488713E-04	0.652702E-04
3	0.237610E-03	0.426350E-03	0.237610E-03	0.316883E-03
4	0.642607E-03	0.104760E-02	0.642607E-03	0.824391E-03
5	0.132081E-02	0.199900E-02	0.132080E-02	0.163535E-02
6	0.230651E-02	0.329221E-02	0.230651E-02	0.277294E-02
7	0.360999E-02	0.491347E-02	0.360999E-02	0.423546E-02
8	0.521792E-02	0.682586E-02	0.521792E-02	0.599767E-02
9	0.709550E-02	0.897307E-02	0.709550E-02	0.801378E-02
10	0.919000E-02	0.112845E-01	0.919000E-02	0.102218E-01
⋮				

45	0.440657E-02	0.534263E-02	0.440660E-02	0.491147E-02
46	0.529242E-02	0.576879E-02	0.529244E-02	0.556998E-02
47	0.575031E-02	0.575473E-02	0.575032E-02	0.579144E-02
48	0.576783E-02	0.532398E-02	0.576783E-02	0.558107E-02
49	0.536517E-02	0.452545E-02	0.536516E-02	0.497521E-02
50	0.459210E-02	0.343772E-02	0.459208E-02	0.403720E-02
51	0.352290E-02	0.215312E-02	0.352286E-02	0.285144E-02

I	XD(I)	XD(I)	XD(I)	XD(I)
2	0.350266E-03	0.700531E-03	0.963263E-02	0.145198E-01
3	0.100825E-02	0.166623E-02	0.277278E-01	0.369691E-01
4	0.191111E-02	0.281397E-02	0.525573E-01	0.653292E-01
5	0.298127E-02	0.405142E-02	0.819876E-01	0.972501E-01
6	0.413276E-02	0.528425E-02	0.113655E+00	0.130237E+00
7	0.527791E-02	0.642306E-02	0.145147E+00	0.161841E+00
8	0.633377E-02	0.738963E-02	0.174185E+00	0.189831E+00
9	0.722794E-02	0.812211E-02	0.198775E+00	0.212348E+00
10	0.790330E-02	0.857866E-02	0.217348E+00	0.228023E+00
⋮				
45	0.395372E-02	0.234794E-02	0.108730E+00	0.870512E-01
46	0.246078E-02	0.745291E-03	0.676726E-01	0.441850E-01
47	0.861748E-03	-0.833375E-03	0.236977E-01	0.197140E-03
48	-0.721946E-03	-0.227447E-02	-0.198553E-01	-0.416597E-01
49	-0.217591E-02	-0.347992E-02	-0.598408E-01	-0.784465E-01
50	-0.340079E-02	-0.437360E-02	-0.935257E-01	-0.107736E+00
51	-0.431663E-02	-0.490573E-02	-0.118769E+00	-0.127765E+00

4.55 $x(t) = \frac{1}{m\omega_d} \int_0^t F(\tau)\, e^{-\gamma\omega_n(t-\tau)} \sin \omega_d(t-\tau)\, d\tau$

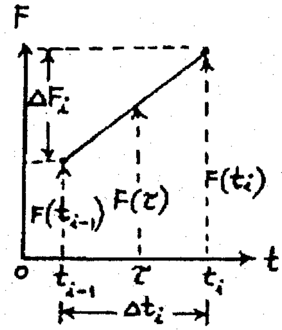
But $\sin \omega_d(t-\tau) = \sin \omega_d t \cos \omega_d \tau - \cos \omega_d t \sin \omega_d \tau$

$x(t) = \{A(t) \cdot \sin \omega_d t - B(t) \cdot \cos \omega_d t\} \frac{e^{-\gamma\omega_n t}}{m\omega_d}$

where

$A(t) = \int_0^t F(\tau)\, e^{\gamma\omega_n\tau} \cos \omega_d \tau \cdot d\tau$

$B(t) = \int_0^t F(\tau)\, e^{\gamma\omega_n\tau} \sin \omega_d \tau \cdot d\tau$



Let $F(\tau)$ be taken as a piecewise linear function during (t_{i-1}, t_i) as

$F(\tau) = F_{i-1} + \left(\frac{\tau - t_{i-1}}{t_i - t_{i-1}}\right)(F_i - F_{i-1})$

We can write

$$A(t_i) = A(t_{i-1}) + \int_{t_{i-1}}^{t_i} \left(\frac{\tau}{\Delta t_i} \Delta F_i \right) e^{\zeta \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$+ \int_{t_{i-1}}^{t_i} \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) e^{\zeta \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$= A(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i} P_1 + \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) P_2$$

where

$$P_1 = \int_{t_{i-1}}^{t_i} \tau \, e^{\zeta \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$P_2 = \int_{t_{i-1}}^{t_i} e^{\zeta \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$B(t_i) = B(t_{i-1}) + \int_{t_{i-1}}^{t_i} \left(\frac{\tau}{\Delta t_i} \Delta F_i \right) e^{\zeta \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

$$+ \int_{t_{i-1}}^{t_i} \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) e^{\zeta \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

$$= B(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i} P_4 + \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) P_3$$

where

$$P_3 = \int_{t_{i-1}}^{t_i} e^{\zeta \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

$$P_4 = \int_{t_{i-1}}^{t_i} \tau \, e^{\zeta \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

Assuming $A(t_1 = 0) = B(t_1 = 0) = 0$,

$$x(t_i) = \frac{e^{-\zeta \omega_n t_i}}{m \omega_d} \left[A(t_i) \sin \omega_d t_i - B(t_i) \cos \omega_d t_i \right]$$

The integrals in P_1, P_2, P_3 and P_4 can be evaluated in closed form.

The computer program and output are given.

```
C ======================================================================
C
C PROBLEM 4.55
C NUMERICAL INTEGRATION OF DUHAMEL INTEGRAL
C
C ======================================================================
C PROBLEM-DEPENDENT DATA
      DIMENSION F(21),X(21),DELT(21),T(21),A(21),B(21)
      NP=21
      XAI=0.1
      OMN=1.0
```

```
      XM=1.0
      DATA F/1.0,.8436,.6910,.5460,.4122,.2929,.1910,.1090,
     2 .04894,.01231,.0,.0,.0,.0,.0,.0,.0,.0,.0,.0,.0/
      DO 10 I=1,21
 10   DELT(I)=0.31416
C END OF PROBLEM-DEPENDENT DATA
      A(1)=0.0
      B(1)=0.0
      OMD=OMN*SQRT(1.0-XAI**2)
      T(1)=0.0
      DO 20 I=2,NP
      T(I)=T(I-1)+DELT(I)
      TIME=T(I)
      CALL PI1(TIME,XAI,OMN,OMD,PP1)
      CALL PI2(TIME,XAI,OMN,OMD,PP2)
      CALL PI3(TIME,XAI,OMN,OMD,PP3)
      CALL PI4(TIME,XAI,OMN,OMD,PP4)
      TIME=T(I-1)
      CALL PI1(TIME,XAI,OMN,OMD,PM1)
      CALL PI2(TIME,XAI,OMN,OMD,PM2)
      CALL PI3(TIME,XAI,OMN,OMD,PM3)
      CALL PI4(TIME,XAI,OMN,OMD,PM4)
      P1=PP1-PM1
      P2=PP2-PM2
      P3=PP3-PM3
      P4=PP4-PM4
      DELF=F(I)-F(I-1)
      A(I)=A(I-1)+(DELF/DELT(I))*P1+(F(I-1)-T(I-1)*DELF/DELT(I))*P2
      B(I)=B(I-1)+(DELF/DELT(I))*P4+(F(I-1)-T(I-1)*DELF/DELT(I))*P3
      X(I)=(EXP(-XAI*OMN*T(I))/(XM*OMD))*(A(I)*SIN(OMD*T(I))-
     2 B(I)*COS(OMD*T(I)))
 20   CONTINUE
      PRINT 30
 30   FORMAT (//,2X,41H NUMERICAL EVALUATION OF DUHAMEL INTEGRAL,
     2 //,5X,2H I,6X,5H T(I),10X,5H F(I),10X,5H X(I),/)
      DO 40 I=2,NP
 40   PRINT 50, I,T(I),F(I),X(I)
 50   FORMAT (2X,I5,3E15.8)
      STOP
      END
C ====================================================================
C
C SUBROUTINE PI1
C
C ====================================================================
      SUBROUTINE PI1 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(T*EXP(XAI*OMN*T)/DEN)*(XAI*OMN*COS(OMD*T)+OMD*SIN(OMD*T))
     2 -(EXP(XAI*OMN*T)/(DEN**2))*(((XAI*OMN)**2-OMD**2)*COS(OMD*T)
     3 +2.0*XAI*OMN*OMD*SIN(OMD*T))
      RETURN
      END
```

```
C ===================================================================
C
C SUBROUTINE PI2
C
C ===================================================================
      SUBROUTINE PI2 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(EXP(XAI*OMN*T)/DEN)*(XAI*OMN*COS(OMD*T)+OMD*SIN(OMD*T))
      RETURN
      END
C ===================================================================
C
C SUBROUTINE PI3
C
C ===================================================================
      SUBROUTINE PI3 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(EXP(XAI*OMN*T)/DEN)*(XAI*OMN*SIN(OMD*T)-OMD*COS(OMD*T))
      RETURN
      END
C ===================================================================
C
C SUBROUTINE PI4
C
C ===================================================================
      SUBROUTINE PI4 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(T*EXP(XAI*OMN*T)/DEN)*(XAI*OMN*SIN(OMD*T)-OMD*COS(OMD*T))
     2 -(EXP(XAI*OMN*T)/(DEN**2))*(((XAI*OMN)**2-OMD**2)*SIN(OMD*T)
     3 -2.0*XAI*OMN*OMD*COS(OMD*T))
      RETURN
      END
```

NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

I	T(I)	F(I)	X(I)
2	0.31415999E+00	0.84359998E+00	0.45415938E-01
3	0.62831998E+00	0.69099998E+00	0.16377741E+00
4	0.94247997E+00	0.54600000E+00	0.32499743E+00
5	0.12566404E+01	0.41219997E+00	0.49747676E+00
6	0.15708008E+01	0.29290003E+00	0.65152758E+00
7	0.18849611E+01	0.19099998E+00	0.76239210E+00
8	0.21991215E+01	0.10900003E+00	0.81256396E+00
9	0.25132818E+01	0.48939999E-01	0.79324198E+00
10	0.28274422E+01	0.12309998E-01	0.70482814E+00
11	0.31416025E+01	0.00000000E+00	0.55646867E+00
12	0.34557629E+01	0.00000000E+00	0.36454141E+00
13	0.37699232E+01	0.00000000E+00	0.14971566E+00
14	0.40840836E+01	0.00000000E+00	-0.66231847E-01
15	0.43982439E+01	0.00000000E+00	-0.26274461E+00
16	0.47124043E+01	0.00000000E+00	-0.42236131E+00
17	0.50265646E+01	0.00000000E+00	-0.53218621E+00
18	0.53407249E+01	0.00000000E+00	-0.58483124E+00
19	0.56548853E+01	0.00000000E+00	-0.57878375E+00
20	0.59690456E+01	0.00000000E+00	-0.51819199E+00
21	0.62832060E+01	0.00000000E+00	-0.41212624E+00

(4.56) The computer program of problem 4.55 can be used to find the relative displacement $z(t)$ of the water tank provided $-m\ddot{y}(\tau)$ is used in place of $F(\tau)$.

Here $\zeta = 0$, $\omega_n = 22.3607$ rad/sec and

$$F(\tau) = -10000 \times 9.81 \times \ddot{y}(\tau) \quad \text{if } \ddot{y} \text{ is in } g\text{'s}.$$

The problem-dependent data for the program of problem 4.55 and the output are given below.

```
C ==================================================================
C
C PROBLEM 4.56
C
C ==================================================================
C PROBLEM-DEPENDENT DATA
      DIMENSION F(15),X(15),DELT(15),T(15),A(15),B(15)
      NP=15
      XAI=0.0
      OMN=22.3607
      XM=10000.0
      DATA F/.0,.45,-.8,-.9,-.6,-.75,-.7,.55,1.75,1.65,.25,
     2 -1.1,-1.4,-1.05,.0/
      DO 10 I=1,15
 10   DELT(I)=0.025
      DO 11 I=1,15
 11   F(I)=-XM*9.81*F(I)
C END OF PROBLEM-DEPENDENT DATA
```

```
NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

    I        T(I)              F(I)               X(I)

    2 0.24999999E-01-0.44144996E+05-0.45271125E-03
    3 0.49999997E-01 0.78480000E+05-0.17453337E-02
    4 0.74999988E-01 0.88290000E+05 0.11150776E-02
    5 0.99999964E-01 0.58860004E+05 0.86095147E-02
    6 0.12499994E+00 0.73575000E+05 0.17519452E-01
    7 0.14999992E+00 0.68670000E+05 0.25374368E-01
    8 0.17499989E+00-0.53955000E+05 0.28478131E-01
    9 0.19999987E+00-0.17167500E+06 0.19676924E-01
   10 0.22499985E+00-0.16186494E+06-0.42601228E-02
   11 0.24999982E+00-0.24525000E+05-0.35448216E-01
   12 0.27499980E+00 0.10791006E+06-0.57387743E-01
   13 0.29999977E+00 0.13734000E+06-0.56341648E-01
   14 0.32499975E+00 0.10300506E+06-0.30433871E-01
   15 0.34999973E+00 0.00000000E+00 0.10307007E-01
```

4.57 The problem-dependent data (to be used in the program of Problem 4.55) and output are given.

```
C PROBLEM-DEPENDENT DATA
      DIMENSION F(30),X(30),DELT(30),T(30),A(30),B(30)
      NP=30
      XAI=0.0
      OMN=8.660254
      XM=2.0
      DATA F /60.0,60.0,60.0,60.0,60.0,100.0,100.0,100.0,100.0,100.0,
     2 30.0,30.0,30.0,30.0,30.0,30.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
     3 0.0,0.0,0.0,0.0,0.0,0.0/
      DO 10 I=1,30
 10   DELT(I)=0.01
C END OF PROBLEM-DEPENDANT DATA
```

NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

I	T(I)	F(I)	X(I)
2	0.99999998E-02	0.60000000E+02	0.14990796E-02
3	0.20000000E-01	0.60000000E+02	0.59849964E-02
4	0.29999999E-01	0.60000000E+02	0.13424230E-01
5	0.39999999E-01	0.60000000E+02	0.23760952E-01
6	0.50000001E-01	0.10000000E+03	0.37250735E-01
7	0.60000002E-01	0.10000000E+03	0.55125091E-01
8	0.70000000E-01	0.10000000E+03	0.77583194E-01
9	0.79999998E-01	0.10000000E+03	0.10445669E+00
10	0.89999996E-01	0.10000000E+03	0.13554408E+00
11	0.99999994E-01	0.30000000E+02	0.17002963E+00
12	0.10999999E+00	0.30000000E+02	0.20532255E+00
13	0.11999999E+00	0.30000000E+02	0.24057558E+00
14	0.13000000E+00	0.30000000E+02	0.27552453E+00
15	0.14000000E+00	0.30000000E+02	0.30990744E+00
16	0.15000001E+00	0.30000000E+02	0.34346649E+00
17	0.16000001E+00	0.00000000E+00	0.37570029E+00
18	0.17000002E+00	0.00000000E+00	0.40536803E+00
19	0.18000002E+00	0.00000000E+00	0.43199742E+00
20	0.19000003E+00	0.00000000E+00	0.45538884E+00
21	0.20000003E+00	0.00000000E+00	0.47536695E+00
22	0.21000004E+00	0.00000000E+00	0.49178207E+00
23	0.22000004E+00	0.00000000E+00	0.50451112E+00
24	0.23000005E+00	0.00000000E+00	0.51345861E+00
25	0.24000005E+00	0.00000000E+00	0.51855767E+00
26	0.25000006E+00	0.00000000E+00	0.51976985E+00
27	0.26000005E+00	0.00000000E+00	0.51708633E+00
28	0.27000004E+00	0.00000000E+00	0.51052707E+00
29	0.28000003E+00	0.00000000E+00	0.50014120E+00
30	0.29000002E+00	0.00000000E+00	0.48600671E+00

(4.58) (i) Find natural frequency:

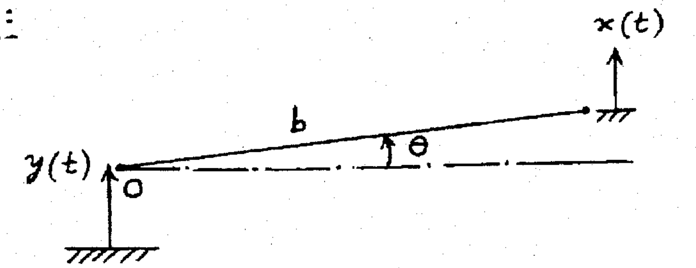
$$J_o = m b^2 = 9 m a^2$$

$$\frac{1}{2} k_t \theta^2 = \frac{1}{2} k (a\theta)^2 = \frac{1}{2} k a^2 \theta^2 \Rightarrow k_t = k a^2$$

$$\omega_n = \sqrt{\frac{k_t}{J_o}} = \sqrt{\frac{k a^2}{9 m a^2}} = \frac{1}{3}\sqrt{\frac{k}{m}} = 2\pi(10) = 20\pi \ \frac{\text{rad}}{\text{sec}} \qquad (E_1)$$

(ii) Find equation of motion:

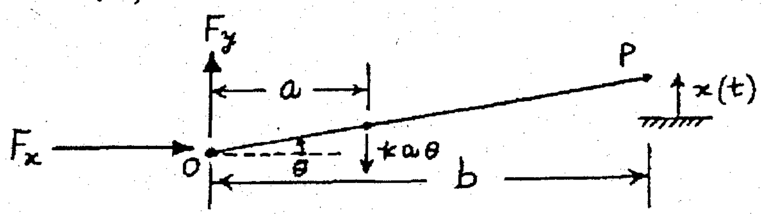
When the base and hence the pivot O is displaced by $y(t)$, the displacement of mass m, $x(t)$, is given by



$$x(t) = y(t) + b\,\theta(t) \qquad (E_2)$$

Relative displacement of mass is

$$z(t) = x(t) - y(t) = b\,\theta(t) \qquad (E_3)$$



Equation of motion:

Σ Forces along y (vertical) direction $= 0 \Rightarrow F_y - k a \theta = m\ddot{x} \qquad (E_4)$

Σ Moments about P $= 0 \qquad \Rightarrow F_y \cdot b - k a \theta (b - a) = 0 \qquad (E_5)$

Solve (E_4) for F_y and substitute the result in (E_5):

$$(k a \theta + m\ddot{x})\, b - k a \theta (b - a) = 0$$

i.e.,
$$m b \ddot{x} + \theta k a^2 = 0 \qquad (E_6)$$

But $\quad x = y + z \quad$ and $\quad \theta(t) = \frac{1}{b} z(t)$

Hence (E_6) becomes

$$m b \ddot{z} + \frac{k a^2}{b} z = - m b \ddot{y}$$

or
$$m \ddot{z} + \frac{k a^2}{b^2} z = - m \ddot{y} \qquad (E_7)$$

Eq. (E_7) can be compared with a standard forced vibration equation for an undamped system:

$$\underset{\sim}{m}\, \ddot{x} + \underset{\sim}{k}\, x = \underset{\sim}{F}(t) \qquad (E_8)$$

Comparison of (E_7) and (E_8) shows that

$$x = z, \quad \underset{\sim}{m} = m, \quad \underset{\sim}{k} = \frac{k a^2}{b^2} \quad \text{and} \quad \underset{\sim}{F} = -m\, \ddot{y} \qquad (E_9)$$

(iii) Find solution:

Solution of Eq. (E_8) under a rectangular pulse is given in problem 4.19. Hence the solution of problem 4.19 can be used to find the relative displacement.

(iv) Design:

Eq. (E_1) gives $\sqrt{k} = 60\pi \sqrt{m}$ or $k = 3600\, \pi^2 m \qquad (E_{10})$

Following procedure can be used to solve the problem:

a. Assume a value of a.

b. Assume a small value of m.

c. Find k from Eq. (E_{10}).

d. Evaluate the solution numerically as outlined in part (iii).

e. If the maximum relative displacement is larger than or equal to 0.02 m, the design is complete.

f. Otherwise, increase the value of m and go to step c.

g. If necessary change the value of a in step a.

(4.59) Let k and m denote the equivalent stiffness and mass of the cutting head.

Equation of motion is: $m\ddot{x} + k x = F(t) \qquad (E_1)$

where $F(t)$ is given by Figs. 4.34 (a) and (b).

Solution of Eq. (E_1) under the force given by Fig. 4.56 (a) can be obtained as in problem 4.15. Solution of (E_1) under the force of Fig. 4.56 (b) can be determined using a procedure similar to that of problem 4.24.

The values of k and m can be determined as follows:

a. Assume a small value for m.

b. Assume a small value for k.

c. Evaluate the response under $F(t)$ given by Fig. 4.56(a).

d. Evaluate the response under F(t) given by Fig. 4.56 (b).

e. If the responses in steps c and d are approximately equal to 0.1 mm and 0.05 mm, the current values of m and k are the desired values.

f. Otherwise, increment the value of m and/or k and go to step c.

4.60 Model the system as a single d.o.f. torsional system with:

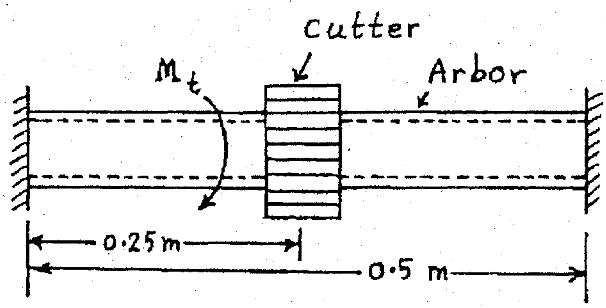
$J_0 = 0.1$ N–m^2 ; $c_t = 0$ (no damping assumed) ;

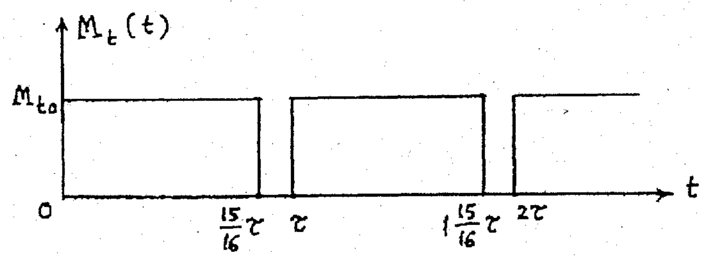
$$k_t = \frac{G\,J}{\ell} = \frac{1}{0.25}\,(80\,(10^9))\left(\frac{\pi}{32}\,(d_o^4 - d_i^4)\right)$$

where d_o = outer diameter of shaft, and d_i = inner diameter.

$$k_t = 62.832\,(10^9)\,(d_o^4 - d_i^4) \tag{1}$$

Torque acting on the arbor due to breakage of one tooth can be modeled as shown in figure.





M_{t0} = 500 N-m, $\tau = \frac{2\,\pi}{\omega} = \frac{2\,\pi}{N\,(2\,\pi)/60} = 60/N = 60/1000 = 0.06$ sec where N = speed of cutter = 1000 rpm. Express $M_t(t)$ in Fourier series:

$$M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\,\omega\,t + b_n \sin n\,\omega\,t\right) \tag{2}$$

(see solution of Problem 4.6 for procedure). where $\omega = \frac{2\,\pi\,N}{60} = 104.72$ rad/sec.

Equation of motion:

$$J_0\,\ddot{\theta} + k_t\,\theta = M_t(t) \tag{3}$$

where $M_t(t)$ is given by Eq. (2). Find solution of Eq. (3) and determine the maximum value of θ, θ_{max}. This value must be less than 1°.

Note:
The solution requires an iterative procedure. Assume values for d_i and d_o. Compute k_t, find solution of $\theta(t)$, and the value of θ_{max}. If θ_{max} exceeds 1°, choose a different set of values for d_i and d_o and continue the process until θ_{max} comes out to be less than 1°.

Chapter 5

Two Degree of Freedom Systems

5.1 Equations of motion
$$\left.\begin{aligned} m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 &= 0 \\ m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 &= 0 \end{aligned}\right\} \qquad (E_1)$$

With $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1, 2$, Eqs. (E_1) give the frequency equation

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{vmatrix} = 0$$

or
$$\omega^4 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right) \omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0 \qquad (E_2)$$

Roots of Eq. (E_2) are

$$\omega_1^2, \omega_2^2 = \frac{k_1 + k_2}{2 m_1} + \frac{k_2}{2 m_2} \mp \sqrt{\frac{1}{4} \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1 k_2}{m_1 m_2}} \qquad (E_3)$$

If $\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} = r_1 X_1^{(1)} \end{Bmatrix}$ and $\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} = r_2 X_1^{(2)} \end{Bmatrix}$,

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1 \omega_1^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2 \omega_1^2 + k_2} \qquad (E_4)$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1 \omega_2^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2 \omega_2^2 + k_2} \qquad (E_5)$$

General solution of (E_1) is

$$\left.\begin{aligned} x_1(t) &= X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ x_2(t) &= r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{aligned}\right\} \qquad (E_6)$$

where $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 and ϕ_2 can be found using Eqs. (5.18).

For $m_1 = m$, $m_2 = 2m$, $k_1 = k$ and $k_2 = 2k$, (E_3) gives

$$\omega_1^2 = (2 - \sqrt{3}) \frac{k}{m} \quad , \quad \omega_2^2 = (2 + \sqrt{3}) \frac{k}{m} \qquad (E_7)$$

When $k = 1000$ N/m and $m = 20$ kg,

$\omega_1 = 3.6603$ rad/sec and $\omega_2 = 13.6603$ rad/sec

$$r_1 = \frac{k_2}{-m_2\omega_1^2 + k_2} = 1.36604, \quad r_2 = \frac{k_2}{-m_2\omega_2^2 + k_2} = -0.36602$$

With $x_1(0) = 1$, $\dot{x}_1(0) = 0$, $x_2(0) = -1$ and $\dot{x}_2(0) = 0$, Eqs. (5.18) give

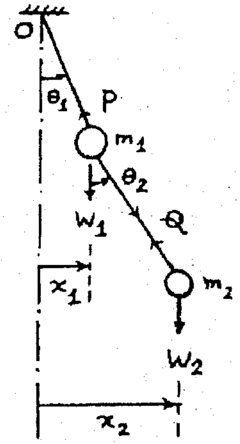
$$X_1^{(1)} = -0.36602, \quad X_1^{(2)} = -1.36603, \quad \phi_1 = 0, \quad \phi_2 = 0$$

Response of the system is

$$x_1(t) = -0.36602 \cos 3.6603\,t - 1.36603 \cos 13.6603\,t$$

$$x_2(t) = -0.5 \cos 3.6603\,t + 0.5 \cos 13.6603\,t$$

5.2 Taking moments about O and mass m_1,



$$m_1 l_1^2 \ddot{\theta}_1 = -W_1(l_1 \sin\theta_1) + Q \sin\theta_2 (l_1 \cos\theta_1) - Q\cos\theta_2 (l_1 \sin\theta_1)$$

$$= -W_1 l_1 \theta_1 + W_2 l_1 (\theta_2 - \theta_1) \qquad (E_1)$$

assuming $Q \simeq W_2$.

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 (l_1 \ddot{\theta}_1) = -W_2 (l_2 \sin\theta_2)$$

$$= -W_2 l_2 \theta_2 \qquad (E_2)$$

Using the relations $\theta_1 = \dfrac{x_1}{l_1}$ and $\theta_2 = \dfrac{x_2 - x_1}{l_2}$, Eqs. (E_1) and (E_2) become

$$m_1 l_1 \ddot{x}_1 + \left[W_1 + W_2\left(\frac{l_1 + l_2}{l_2}\right)\right] x_1 - \frac{W_2 l_1}{l_2} x_2 = 0 \qquad (E_3)$$

$$m_2 l_2 \ddot{x}_2 - W_2 x_1 + W_2 x_2 = 0 \qquad (E_4)$$

When $m_1 = m_2 = m$, $l_1 = l_2 = l$ and $W_1 = W_2 = mg$, Eqs. (E_3) and (E_4) give

$$\left.\begin{aligned} ml\,\ddot{x}_1 + 3mg\,x_1 - mg\,x_2 &= 0 \\ ml\,\ddot{x}_2 - mg\,x_1 + mg\,x_2 &= 0 \end{aligned}\right\} \qquad (E_5)$$

For harmonic motion $x_i(t) = X_i \cos\omega t$; $i = 1,2$, Eqs. (E_5) become

$$\left.\begin{aligned} -\omega^2 ml\,X_1 + 3mg\,X_1 - mg\,X_2 &= 0 \\ -\omega^2 ml\,X_2 - mg\,X_1 + mg\,X_2 &= 0 \end{aligned}\right\} \qquad (E_6)$$

from which the frequency equation can be obtained as

$$\omega^4 m^2 l^2 - (4 m^2 l g)\,\omega^2 + 2 m^2 g^2 = 0$$

i.e. $\quad \omega_1^2, \omega_2^2 = (2 \mp \sqrt{2}) \dfrac{g}{l}$

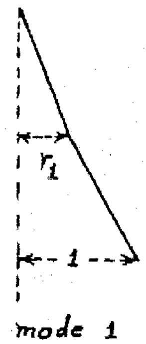
$$\therefore \quad \omega_1 = 0.7654\sqrt{\frac{g}{l}}, \quad \omega_2 = 1.8478\sqrt{\frac{g}{l}}$$

Ratio of amplitudes is given by Eq. (E_6) as

$$\frac{X_1}{X_2} = \frac{m g}{-\omega^2 m l + 3 m g} = \frac{1}{\left(-\omega^2 \frac{l}{g} + 3\right)}$$

In mode 1, $\omega_1 = 0.7654 \sqrt{\frac{g}{l}}$, $r_1 = \left(\frac{X_1}{X_2}\right)^{(1)} = 0.4142$

No node.

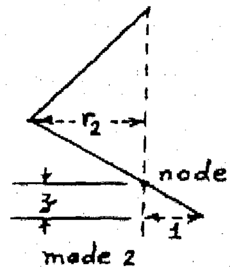


In mode 2, $\omega_2 = 1.8478 \sqrt{\frac{g}{l}}$,

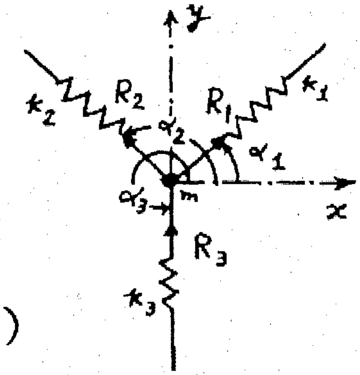
$$r_2 = \left(\frac{X_1}{X_2}\right)^{(2)} = -2.4133$$

one node located at z:

$$\frac{z}{1} = \frac{1-z}{2.4133} \quad \text{or} \quad z = 0.2930$$



5.3 Let R_1, R_2 and R_3 be the restoring forces in springs. Equations of motion of mass m in x and y directions are



$$m\ddot{x} = \sum_{i=1}^{3} R_i \cos\alpha_i \qquad (E_1)$$

$$m\ddot{y} = \sum_{i=1}^{3} R_i \sin\alpha_i \qquad (E_2)$$

where $\quad R_i = -k_i\,(x\cos\alpha_i + y\sin\alpha_i) \qquad (E_3)$

Eqs. (E_1) to (E_3) give

$$m\ddot{x} + \sum_{i=1}^{3} k_i\,(x\cos^2\alpha_i + y\sin\alpha_i\cos\alpha_i) = 0 \qquad (E_4)$$

$$m\ddot{y} + \sum_{i=1}^{3} k_i\,(x\sin\alpha_i\cos\alpha_i + y\sin^2\alpha_i) = 0 \qquad (E_5)$$

For $\alpha_1 = 45°$, $\alpha_2 = 135°$, $\alpha_3 = 270°$ and $k_1 = k_2 = k_3 = k$, Eqs. (E_4) and (E_5) reduce to

$$m\ddot{x} + k x = 0 \qquad (E_6)$$

$$m\ddot{y} + 2k y = 0 \qquad (E_7)$$

These equations are uncoupled. For harmonic motion,
$x(t) = X\cos(\omega t + \phi)$, $\quad y(t) = Y\cos(\omega t + \phi)$, and hence

$\omega_1 = \sqrt{\frac{k}{m}}$ for motion in x direction

$\omega_2 = \sqrt{\frac{2k}{m}}$ for motion in y direction

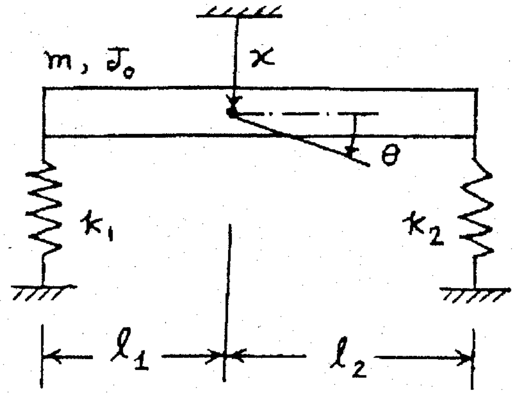
Natural modes are given by

$$x(t) = X\cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right)$$

$$y(t) = Y\cos\left(\sqrt{\frac{2k}{m}}\,t + \phi_2\right)$$

where X, ϕ_1, Y and ϕ_2 can be determined from initial conditions.

(5.4) Equations of motion in terms of x and θ:



$$m\ddot{x} + k_1(x - l_1\theta) + k_2(x + l_2\theta) = 0 \qquad (E_1)$$

$$J_0\ddot{\theta} - k_1 l_1(x - l_1\theta) + k_2 l_2(x + l_2\theta) = 0 \qquad (E_2)$$

For free vibration,

$$x(t) = X \cos(\omega t + \phi) \qquad (E_3)$$

$$\theta(t) = \Theta \cos(\omega t + \phi) \qquad (E_4)$$

and Eqs. (E_1) and (E_2) become

$$\begin{bmatrix} -m\omega^2 + k_1 + k_2 & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & -J_0\omega^2 + k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (E_5)$$

Frequency equation is

$$\begin{vmatrix} -m\omega^2 + k_1 + k_2 & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & -J_0\omega^2 + k_1 l_1^2 + k_2 l_2^2 \end{vmatrix} = 0 \qquad (E_6)$$

i.e.,

$$\begin{vmatrix} -\omega^2 + 5000 & 100 \\ 100 & -0.3\,\omega^2 + 2030 \end{vmatrix} = 0$$

i.e.,

$$0.3\,\omega^4 - 3530\,\omega^2 + 10.14 \times 10^6 = 0$$

i.e.,

$$\omega^2 = 6785.3373, \quad 4981.3293$$

$$\therefore \quad \omega_1 = 70.5785 \text{ rad/sec}, \qquad \omega_2 = 82.3732 \text{ rad/sec}$$

Mode shapes:

$$(-1000\,\omega_1^2 + 5 \times 10^6)\,X + 0.1 \times 10^6\,\Theta = 0$$

or

$$\left.\frac{X}{\Theta}\right|_{\omega_1} = \frac{-0.1 \times 10^6}{-1000\,\omega_1^2 + 5 \times 10^6} = -5.3476$$

and

$$\left.\frac{X}{\Theta}\right|_{\omega_2} = \frac{-0.1 \times 10^6}{-1000\,\omega_2^2 + 5 \times 10^6} = 0.05601$$

(5.5)

$$k_1 = \text{stiffness of girder} = \frac{48\,E\,I}{\ell^3} = \frac{48\,(6\,(10^{12}))}{(30\,(12))^3} = 6.1728\,(10^6) \text{ lb/in}$$

$$k_2 = \text{stiffness of rope} = \frac{A\,E}{\ell} = \frac{A\,(30\,(10^6))}{(20)\,(12)} = 12.5\,(10^4)\,A \text{ lb/in}$$

m_1 = mass of trolley = 8000/386.4 = 20.7039 lb–sec^2/in
m_2 = mass of load = 2000/386.4 = 5.1760 lb–sec^2/in

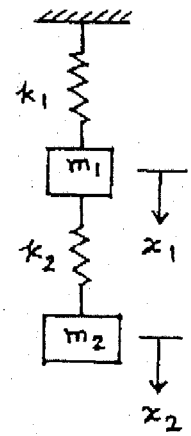
Desired frequency value: $\omega_1 > 20$ Hz. Let $\omega_1 = 25$ Hz $= 157.08$ rad/sec or $\omega_1^2 = 24674.1264$ (rad/sec)^2.

Fundamental natural frequency is given by (see Eq. (E3) in solution of Problem 5.1):

$$\omega_1^2 = \frac{k_1 + k_2}{2\ m_1} + \frac{k_2}{2\ m_2} - \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1\ k_2}{m_1\ m_2}} \qquad (1)$$

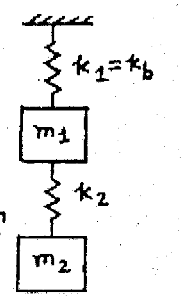
Using the known values of k_1, m_1, and m_2, a series of trial values of A are given and Eq. (1) is evaluated to find the corresponding values of ω_1^2. The results are given in the table below. It can be seen that A = 1.1 in^2 yields a frequency that satisfies the specification.

A (in^2)	k_2 (lb/in)	ω_1^2 (rad/sec)^2	ω_1 (rad/sec)
0.6	0.7500 (10^5)	0.1434 (10^5)	119.7
0.7	0.8750 (10^5)	0.1715 (10^5)	131.0
0.8	0.1000 (10^6)	0.1946 (10^5)	139.5
0.9	0.1125 (10^6)	0.2176 (10^5)	147.5
1.0	0.1250 (10^6)	0.2406 (10^5)	155.1
1.1	0.1375 (10^6)	0.2662 (10^5)	163.2
1.2	0.1500 (10^6)	0.2918 (10^5)	170.8
1.3	0.1625 (10^6)	0.3123 (10^5)	176.7
1.4	0.1750 (10^6)	0.3379 (10^5)	183.8
1.5	0.1875 (10^6)	0.3610 (10^5)	190.0



5.6

$$k_1 = \frac{48EI}{l^3} = \frac{48\,(2.06\times10^{11})(0.02)}{(40)^3} = 3.09\times10^6\ \text{N/m}$$



$k_2 = 0.3\times10^6$ N/m, $m_1 = 1000$ kg, $m_2 = 5000$ kg

Eq. (E_3) in the solution of problem 5.1 gives

$$\omega_1^2,\ \omega_2^2 = \frac{3.39\times10^6}{2000} + \frac{0.3\times10^6}{10\,000} \mp \sqrt{\frac{1}{4}\left(\frac{3.39\times10^6}{1000} + \frac{0.3\times10^6}{5000}\right)^2 - \frac{3.09\times0.3\times10^{12}}{5\times10^6}}$$

$$= (1.725 \mp 1.6704)\,10^3$$

$\omega_1 = 7.3892$ rad/s, $\omega_2 = 58.2701$ rad/s

From Eqs. (E_4) and (E_5) of solution of problem 5.1,

$$r_1 = \frac{k_2}{-m_2\omega_1^2 + k_2} = \frac{0.3\times10^6}{-5000\,(54.6003) + 0.3\times10^6} = 11.1117$$

$$r_2 = \frac{k_2}{-m_2\omega_2^2 + k_2} = \frac{0.3\times10^6}{-5000\,(3395.4046) + 0.3\times10^6} = -0.01799$$

Mode shapes are $\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 11.1117 \end{Bmatrix} X_1^{(1)}$, $\vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ -0.01799 \end{Bmatrix} X_1^{(2)}$

(5.7) Frequency equation:

$$\left| \left[-\omega^2 \, [m] + [k] \right] \right| = 0$$

or

$$\begin{vmatrix} (k_{11} - \omega^2 \, m_1) & k_{12} \\ k_{21} & (k_{22} - \omega^2 \, m_2) \end{vmatrix} = 0 \qquad (1)$$

Expansion of the determinantal equation (1) gives:

$$(m_1 \, m_2) \, \omega^4 - (m_1 \, k_{22} + m_2 \, k_{11}) \, \omega^2 + (k_{11} \, k_{22} - k_{12}^2) = 0 \qquad (2)$$

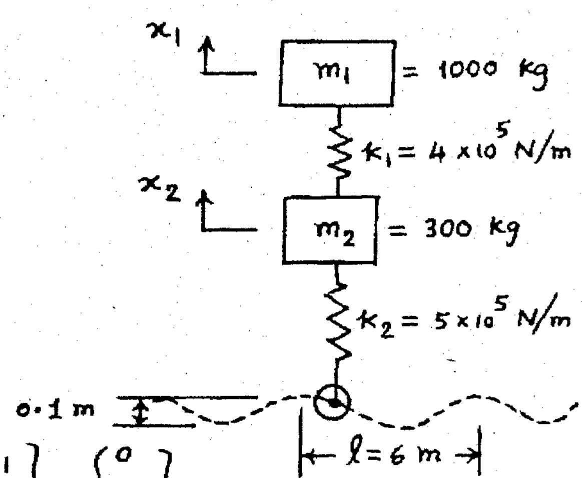
Roots of Eq. (2):

$$\omega_2^2, \, \omega_1^2 = \frac{(m_1 \, k_{22} + m_2 \, k_{11}) \pm \sqrt{\left\{ (m_1 \, k_{22} - m_2 \, k_{11})^2 + 4 \, m_1 \, m_2 \, k_{12}^2 \right\}}}{2 \, m_1 \, m_2} \qquad (3)$$

Substitution of known expressions for k_{11}, k_{12}, and k_{22} into Eq. (3) yields:

$$\omega_2^2, \, \omega_1^2 = \frac{48}{7} \, \frac{E \, I}{m_1 \, m_2} \left[(m_1 + 8 \, m_2) \pm \sqrt{(m_1 - 8 \, m_2)^2 + 25 \, m_1 \, m_2} \right] \qquad (4)$$

(5.8) Equations of motion:



$$\left. \begin{aligned} m_1 \ddot{x}_1 + k_1 x_1 - k_1 x_2 &= 0 \\ m_2 \ddot{x}_2 + (k_1 + k_2) x_2 - k_1 x_1 &= 0 \end{aligned} \right\} \quad (E_1)$$

Let $x_i(t) = X_i \cos(\omega t + \phi)$;

$$i = 1, 2 \qquad (E_2)$$

Eq. (E_1) becomes

$$\begin{bmatrix} -m_1 \omega^2 + k_1 & -k_1 \\ -k_1 & -m_2 \omega^2 + k_1 + k_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$\begin{vmatrix} -m_1 \omega^2 + k_1 & -k_1 \\ -k_1 & -m_2 \omega^2 + k_1 + k_2 \end{vmatrix} = 0$$

i.e.,

$$m_1 m_2 \omega^4 - (m_1 k_1 + m_1 k_2) \omega^2 - k_1 m_2 \omega^2 + k_1 k_2 = 0$$

i.e.,

$$\omega^2 = \Big[(m_1 k_1 + m_1 k_2 + m_2 k_1) \pm \left(m_1^2 k_1^2 + m_1^2 k_2^2 + m_2^2 k_1^2 + 2 m_1^2 k_1 k_2 - 2 m_1 m_2 k_1 k_2 + 2 m_1 m_2 k_1^2\right)^{\frac{1}{2}} \Big] \Big/ 2 m_1 m_2 \qquad (E_3)$$

Since $m_1 = 1000$ kg, $m_2 = 300$ kg, $k_1 = 4\times10^5$ N/m and $k_2 = 5\times10^5$ N/m, Eq. (E_3) gives

$\omega_1 = 14.4539$ rad/sec , $\omega_2 = 56.4897$ rad/sec

$$f_1 = \frac{14.4539}{2\pi} \text{ Hz} = \frac{s_1\,(1000)}{3600}\left(\frac{1}{l}\right) = \frac{s_1}{21.6}$$

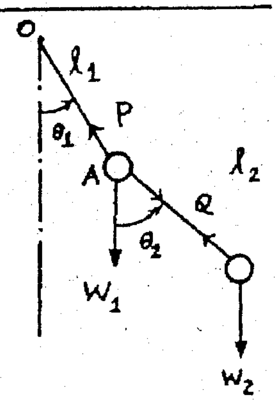
where $l = 6$ m and s_1 is in km/hr.

$$\therefore\ s_1 = \text{critical velocity \# 1} = \frac{14.4539}{2\pi}(21.6) = 49.6887 \text{ km/hr}$$

$$f_2 = \frac{56.4897}{2\pi} \text{ Hz} = s_2\left(\frac{1000}{3600}\right)\left(\frac{1}{l}\right) = \frac{s_2}{21.6}$$

$$\therefore\ s_2 = \text{critical velocity \# 2} = \frac{56.4897}{2\pi}(21.6) = 194.1968 \text{ km/hr.}$$

(5.9) Equations of motion for rotation about o and A give



$$m_1 l_1^2 \ddot{\theta}_1 = -W_1 l_1 \sin\theta_1 + Q \sin\theta_2 (l_1 \cos\theta_1) - Q\cos\theta_2 (l_1 \sin\theta_1)$$

$$= -W_1 l_1 \theta_1 + Q\, l_1 (\theta_2 - \theta_1)$$

$$= -W_1 l_1 \theta_1 + W_2\, l_1 (\theta_2 - \theta_1) \quad ---- \quad (E_1)$$

since $Q \simeq W_2$.

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 (l_1 \ddot{\theta}_1) = -W_2 l_2 \sin\theta_2$$

$$= -W_2 l_2 \theta_2 \quad ---- \quad (E_2)$$

For $m_1 = m_2 = m$ and $l_1 = l_2 = l$, Eqs. (E_1) and (E_2) become

$$m l \ddot{\theta}_1 + 2mg\,\theta_1 - mg\,\theta_2 = 0$$

$$m l \ddot{\theta}_1 + m l \ddot{\theta}_2 + mg\,\theta_2 = 0$$

Assuming $\theta_i(t) = \Theta_i \cos(\omega t + \phi)$; $i = 1,2$, we get

$$\begin{bmatrix} -\omega^2 m l + 2mg & -mg \\ -\omega^2 m l & -\omega^2 m l + mg \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad ---- (E_3)$$

Defining $\lambda = \dfrac{\omega^2 m l}{mg} = \dfrac{\omega^2 l}{g}$, frequency equation can be obtained as

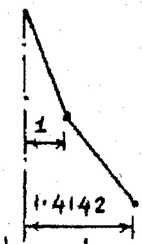
$$\begin{vmatrix} -\lambda + 2 & -1 \\ -\lambda & -\lambda + 1 \end{vmatrix} = \lambda^2 - 4\lambda + 2 = 0$$

$\lambda_1 = 2 - \sqrt{2} = 0.5858$, $\lambda_2 = 2 + \sqrt{2} = 3.4142$

$\omega_1 = 0.7654 \sqrt{\frac{g}{l}}$, $\omega_2 = 1.8478 \sqrt{\frac{g}{l}}$

For ω_1, first equation in (E_3) gives for $\Theta_1 = 1$,

$$\Theta_2 = -\lambda_1 + 2 = \sqrt{2} = 1.4142$$



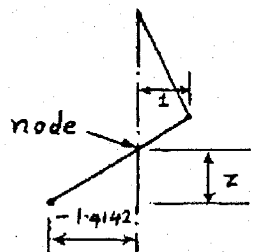
First mode

For ω_2, first equation in (E_3) gives for $\Theta_1 = 1$,

$$\Theta_2 = -\lambda_2 + 2 = -\sqrt{2} = -1.4142$$

Location of node :

$$\frac{z}{1.4142} = \frac{1-z}{1} \; ; \quad z = 0.4142$$



Second mode

(5.10) Eq. (E_3) in the solution of problem 5.1 gives

$$\omega_1^2, \omega_2^2 = \frac{k_1+k_2}{2m_1} + \frac{k_2}{2m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$

For $m_1 = m_2 = m$ and $k_1 = k_2 = k$, we get

$$\omega_1^2, \omega_2^2 = \frac{k}{m} + \frac{k}{2m} \mp \sqrt{\frac{1}{4}\left(\frac{3k}{m}\right)^2 - \frac{k^2}{m^2}} = \frac{3k}{2m} \mp \frac{k}{m}\sqrt{\frac{5}{4}}$$

$$\omega_1^2 = 0.382 \frac{k}{m} , \quad \omega_2^2 = 2.618 \frac{k}{m}$$

$$\omega_1 = 0.6181 \sqrt{\frac{k}{m}} , \quad \omega_2 = 1.6180 \sqrt{\frac{k}{m}}$$

Mode shapes are given by (see Eqs. (E_4) and (E_5) of problem 5.1)

$$r_1 = \frac{k_2}{-m_2\omega_1^2 + k_2} = 1.6181 \; ; \quad \vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.6181 \end{Bmatrix} X_1^{(1)}$$

$$r_2 = \frac{k_2}{-m_2\omega_2^2 + k_2} = -0.6180 \; ; \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -0.6180 \end{Bmatrix} X_1^{(2)}$$

(5.11) For the system of Fig. 5.3 (a),

$$\vec{X}^{(1)T} [m] \vec{X}^{(2)} = \{ X_1^{(1)} \;\; X_2^{(1)} \} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = X_1^{(1)} X_1^{(2)} \{1 \;\; r_1\} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} 1 \\ r_2 \end{Bmatrix}$$

$$= X_1^{(1)} X_1^{(2)} (m_1 + m_2 r_1 r_2)$$

But

$$m_1 + m_2 r_1 r_2 = m_1 + m_2 \left(\frac{k_2}{-m_2\omega_1^2 + k_2 + k_3}\right)\left(\frac{k_2}{-m_2\omega_2^2 + k_2 + k_3}\right) \equiv \frac{N}{D}$$

where

$N = m_1(-m_2\omega_1^2 + k_2 + k_3)(-m_2\omega_2^2 + k_2 + k_3) + m_2 k_2^2$, and

$D = (-m_2\omega_1^2 + k_2 + k_3)(-m_2\omega_2^2 + k_2 + k_3)$

By substituting

$$\omega_1^2, \omega_2^2 = \frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{2m_1m_2} \mp \frac{1}{2}\left\{\left[\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{m_1m_2}\right]^2 - 4\,\frac{(k_1+k_2)(k_2+k_3)-k_2^2}{m_1m_2}\right\}^{1/2}, \text{ we get}$$

$$N = m_1m_2^2\,\omega_1^2\,\omega_2^2 - m_1m_2\,k_2\,\omega_2^2 - m_1m_2\,k_3\,\omega_2^2 - m_1m_2\,k_2\,\omega_1^2 + m_1\,k_2^2 + m_1k_2k_3 - m_1m_2k_3\,\omega_1^2 + m_1k_2k_3 + m_1k_3^2 + m_2k_2^2$$

$$= m_1m_2^2\left\{\left[\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{2m_1m_2}\right]^2 - \frac{1}{4}\left[\left(\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{m_1m_2}\right)^2 - 4\left(\frac{(k_1+k_2)(k_2+k_3)-k_2^2}{m_1m_2}\right)\right]\right\} - m_1m_2k_2\left\{\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{m_1m_2}\right\}$$

$$- m_1m_2k_3\left\{\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{m_1m_2}\right\} + m_1k_2^2 + m_2k_2^2 + 2m_1k_2k_3 + m_1k_3^2$$

$$= 0$$

5.12 Eq. (5.10) gives

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left\{\frac{(800)1+(700)2}{2}\right\} \mp \frac{1}{2}\left[\left\{\frac{(800)1+(700)2}{2}\right\}^2 - 4\left\{\frac{(800)(700)-(500)^2}{2}\right\}\right]^{1/2}$$

$$= 550 \mp 384.0573 = 165.9427,\ 934.0573$$

$\omega_1 = 12.8819$ rad/s, $\omega_2 = 30.5624$ rad/s

5.13 Eq. (E_3) in the solution of problem 5.1 gives

$$\omega_1^2, \omega_2^2 = \frac{8000}{2} + \frac{6000}{2} \mp \sqrt{\frac{1}{4}\left(\frac{8000}{1}+\frac{6000}{1}\right)^2 - \frac{12\times10^6}{1}} = 917.2,\ 13082.8$$

$\omega_1 = 30.2853$ rad/sec, $\omega_2 = 114.3801$ rad/s

Eqs. (E_4) and (E_5) of solution of problem 5.1 give

$$r_1 = \frac{k_2}{-m_2\omega_1^2 + k_2} = \frac{6000}{-917.2+6000} = 1.1805\ ; \quad \vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.1805 \end{Bmatrix} X_1^{(1)}$$

$$r_2 = \frac{k_2}{-m_2\omega_2^2 + k_2} = \frac{6000}{-13082.8+6000} = -0.8471\ ; \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -0.8471 \end{Bmatrix} X_1^{(2)}$$

5.14 Same as example 5.1 with $m_1 = m_2 = 25$ kg and $k_1 = k_2 = k_3 = 50\,000$ N/m

$$\omega_1 = \sqrt{\frac{k}{m}} = \sqrt{\frac{50\,000}{25}} = 44.7214 \text{ rad/s}$$

$$\omega_2 = \sqrt{\frac{3k}{m}} = \sqrt{\frac{150\,000}{25}} = 77.4597 \text{ rad/s}$$

General motion is given by Eq. (E_{10}) of Example 5.1 :

$$x_1(t) = X_1^{(1)} \cos(44.7214\,t + \phi_1) + X_1^{(2)} \cos(77.4597\,t + \phi_2)$$

$$x_2(t) = X_1^{(1)} \cos(44.7214\,t + \phi_1) - X_1^{(2)} \cos(77.4597\,t + \phi_2)$$

5.15 Eq.(5.10) gives

$$\omega_1^2,\ \omega_2^2 = \frac{1}{2}\left\{\frac{3000(2)+4000(1)}{2}\right\} \mp \frac{1}{2}\left[\left\{\frac{3000(2)+4000(1)}{2}\right\}^2 - 4\left\{\frac{3000(4000)-1000^2}{2}\right\}\right]^{\frac{1}{2}}$$

$$= 1633.9746,\ 3366.0254$$

$\omega_1 = 40.4225$ rad/s, $\omega_2 = 58.0175$ rad/s

$$r_1 = \frac{k_2}{-m_2\omega_1^2 + k_2 + k_3} = \frac{1000}{-2(1633.9746)+4000} = 1.3660$$

$$r_2 = \frac{k_2}{-m_2\omega_2^2 + k_2 + k_3} = \frac{1000}{-2(3366.0254)+4000} = -0.3660$$

When $x_1(0) = x_2(0) = \dot{x}_2(0) = 0$ and $\dot{x}_1(0) = 20$ m/s,

$$X_1^{(1)} = \frac{1}{(-0.366-1.366)}\left[\frac{+0.366\,(20)}{40.4225}\right] = -0.1046$$

$$X_1^{(2)} = \frac{1}{-1.732}\left[\frac{1.366\,(20)}{58.0175}\right] = -0.2719$$

$$\phi_1 = \phi_2 = \tan^{-1}(\infty) = \frac{\pi}{2}$$

Motion of the two masses is given by Eq.(5.15):

$$x_1(t) = 0.1046 \sin 40.4225\,t + 0.2719 \sin 58.0175\,t$$

$$x_2(t) = +(1.3660)(0.1046) \sin 40.4225\,t - (-0.3660)(-0.2719) \sin 58.0175\,t$$

$$= 0.1429 \sin 40.4225\,t - 0.09952 \sin 58.0175\,t$$

5.16 (a) $\omega_1^2 = 917.2$, $\omega_2^2 = 13\,082.8$, $r_1 = 1.1805$, $r_2 = -0.8471$

$x_1(0) = 0.2$, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$

Eq.(5.18) gives

$$X_1^{(1)} = \frac{1}{(-0.8471-1.1805)}\left[(-0.8471)(0.2)\right] = 0.08356$$

$$X_1^{(2)} = \frac{1}{(-0.8471-1.1805)}\left[(-1.1805)(0.2)\right] = 0.11644$$

$$\phi_1 = \phi_2 = \tan^{-1}(0) = 0$$

$$x_1(t) = 0.08356 \cos 30.2853\,t + 0.11644 \cos 114.3801\,t$$

$$x_2(t) = (1.1805)(0.08356) \cos 30.2853\,t + (-0.8471)(0.11644) \cos 114.3801\,t$$

$$= 0.09864 \cos 30.2853\,t - 0.09864 \cos 114.3801\,t$$

(b) $x_1(0) = 0.2$, $x_2(0) = \dot{x}_1(0) = 0$, $\dot{x}_2(0) = 5.0$

Eq.(5.18) gives

$$X_1^{(1)} = \frac{1}{-2.0276}\left[\{-0.8471(0.2)\}^2 + \frac{(5)^2}{917.2}\right]^{1/2} = -0.1167$$

$$x_1^{(2)} = \frac{1}{-2.0276}\left[\left\{-1.1805\,(0.2)\right\}^2 + \frac{(-5)^2}{13082.8}\right]^{1/2} = -0.1184$$

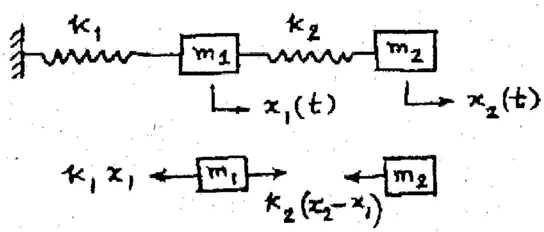
$$\phi_1 = \tan^{-1}\left\{\frac{5.0}{30.2853\quad(-0.8471)(0.2)}\right\} = \tan^{-1}(-0.9745) = 135.7399^\circ$$

$$\phi_2 = \tan^{-1}\left\{\frac{-5.0}{114.3801\,(-1.1805)(0.2)}\right\} = \tan^{-1}(0.1851) = 10.4895^\circ$$

$$x_1(t) = -0.1167\cos(30.2853\,t + 135.7399^\circ) - 0.1184\cos(114.3801\,t + 10.4895^\circ)$$

$$x_2(t) = (1.1805)(-0.1167)\cos(30.2853\,t + 135.7399^\circ) - 0.8471\,(-0.1184)\cos(114.3801\,t + 10.4895^\circ)$$
$$= -0.1378\cos(30.2853\,t + 135.7399^\circ) + 0.1003\cos(114.3801\,t + 10.4895^\circ)$$

5.17 Equivalent system is shown in figure:



$$k_i = 2\left(\frac{24\,EI_i}{h_i^3}\right)\ ;\quad i = 1, 2$$

$$k_1 = k_2 = k = \frac{48\,EI}{h^3}\ ;\ m_1 = 2m,\ m_2 = m$$

Equations of motion: $\quad m_1\ddot{x}_1 + (k_1 + k_2)\,x_1 - k_2 x_2 = 0$

$$m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$$

For harmonic motion $\;x_i(t) = X_i\cos(\omega t + \phi)\ ;\ i = 1, 2$, we get

$$\begin{bmatrix} (-\omega^2 m_1 + k_1 + k_2) & -k_2 \\ -k_2 & (-\omega^2 m_2 + k_2) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad ----(E_1)$$

Frequency equation is

$$\begin{vmatrix} (-\omega^2 m_1 + k_1 + k_2) & -k_2 \\ -k_2 & (-\omega^2 m_2 + k_2) \end{vmatrix} = 0$$

or $\quad \omega^4 m_1 m_2 - \omega^2(m_2 k_1 + m_2 k_2 + m_1 k_2) + k_1 k_2 = 0$

$$\omega^2 = \frac{(m_2 k_1 + m_2 k_2 + m_1 k_2) \pm \sqrt{(m_2 k_1 + m_2 k_2 + m_1 k_2)^2 - 4\,m_1 m_2 k_1 k_2}}{2\,m_1 m_2} \quad ---(E_2)$$

For given data,

$$\omega^2 = \frac{(mk + mk + 2mk) \pm \sqrt{(mk + mk + 2mk)^2 - 8\,m^2 k^2}}{4\,m^2} = \frac{k}{m}\left(1 \pm \frac{1}{\sqrt{2}}\right)$$

$$\omega_1 = 0.5412\sqrt{\frac{k}{m}} = 3.7495\sqrt{\frac{EI}{m h^3}}\ ;\quad \omega_2 = 1.3066\sqrt{\frac{k}{m}} = 9.0524\sqrt{\frac{EI}{m h^3}}$$

From Eq. (E_1), we get

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-\omega_1^2 m_1 + k_1 + k_2}{k_2} = \frac{-2m\,\omega_1^2 + 2k}{k} = \frac{-2(0.2929\,k) + 2k}{k}$$
$$= 1.4142$$
$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-\omega_2^2 m_1 + k_1 + k_2}{k_2} = \frac{-2m\,\omega_2^2 + 2k}{k} = \frac{-2(1.7071\,k) + 2k}{k}$$
$$= -1.4142$$

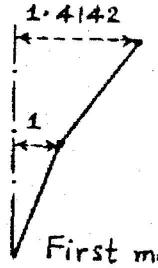
Mode shapes are:

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.4142 \end{Bmatrix} X_1^{(1)}$$

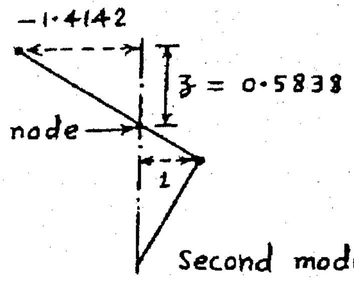
$$\vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ -1.4142 \end{Bmatrix} X_1^{(2)}$$

Location of node:

$$\frac{z}{1.4142} = \frac{1-z}{1} \; ; \quad z = 0.5838$$

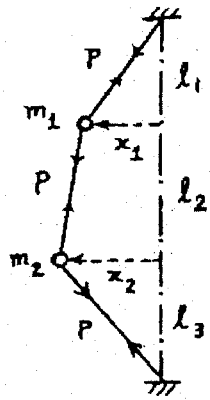


First mode
(no node)



Second mode
(one node)

(5.18) Let P be the tension in the string. Horizontal components of tension (along x_1 direction) in the string lying above and below m_1 are $-\frac{x_1 P}{l_1}$ and $-\frac{(x_1 - x_2)P}{l_2}$, respectively.



Newton's second law gives

$$m_1 \ddot{x}_1 = -\frac{x_1 P}{l_1} - \frac{(x_1 - x_2)P}{l_2} \quad \text{or} \quad m_1 \ddot{x}_1 + \frac{x_1}{l_1} P + \left(\frac{x_1 - x_2}{l_2}\right) P = 0$$

Similarly

$$m_2 \ddot{x}_2 + \frac{x_2}{l_3} P - \left(\frac{x_1 - x_2}{l_2}\right) P = 0$$

With $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1,2$, and $l_1 = l_2 = l_3 = l$, $m_1 = m_2 = m$,

$$\begin{bmatrix} \left(-m\omega^2 + \frac{2P}{l}\right) & -\frac{P}{l} \\ -\frac{P}{l} & \left(-m\omega^2 + \frac{2P}{l}\right) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{----} \; (E_1)$$

This gives the frequency equation

$$\left(-m\omega^2 + \frac{2P}{l}\right)^2 - \left(\frac{P}{l}\right)^2 = \left(-m\omega^2 + \frac{3P}{l}\right)\left(-m\omega^2 + \frac{P}{l}\right) = 0$$

$$\therefore \quad \omega_1 = \sqrt{\frac{P}{ml}} \; , \quad \omega_2 = \sqrt{\frac{3P}{ml}}$$

From first of Eqs. (E_1),

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m\omega_1^2 + \frac{2P}{l}}{\left(\frac{P}{l}\right)} = 1 \; ; \quad r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m\omega_2^2 + \frac{2P}{l}}{\left(\frac{P}{l}\right)} = -1$$

mode shapes are: $\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} X_1^{(1)}$, $\vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} X_1^{(2)}$

No node — one node at middle of the two masses

5.19 For $m_1 = 3m$, $m_2 = m$, $k_1 = 3k$ and $k_2 = k$, Eq. (E_2) in solution of problem 5.17 gives

$$\omega^2 = \frac{(3mk + mk + 3mk) \pm \sqrt{(3mk + mk + 3mk)^2 - 36 k^2 m^2}}{6 m^2} = \frac{k}{6m}\left(7 \pm \sqrt{13}\right)$$

$$\omega^2 = 0.5657 \frac{k}{m}, \quad 1.7676 \frac{k}{m}$$

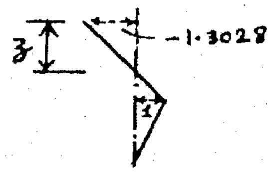
$$\omega_1 = 0.7521 \sqrt{\frac{k}{m}}, \quad \omega_2 = 1.3295 \sqrt{\frac{k}{m}}$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-\omega_1^2 m_1 + k_1 + k_2}{k_2} = \frac{-0.5657(3) + 3 + 1}{1} = 2.3029$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-\omega_2^2 m_1 + k_1 + k_2}{k_2} = \frac{-1.7676(3) + 3 + 1}{1} = -1.3028$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 2.3029 \end{Bmatrix} X_1^{(1)}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ -1.3028 \end{Bmatrix} X_1^{(2)}$$

No node — one node

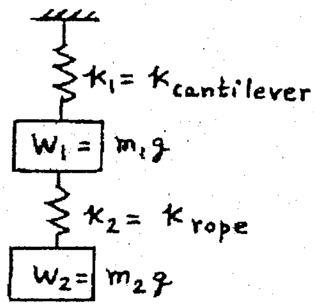


$$\frac{z}{1.3028} = \frac{1 - z}{1}; \quad z = 0.5657$$

5.20

$$k_1 = \frac{3EI}{b^3} = \frac{3E\left(\frac{1}{12} a t^3\right)}{b^3} = \frac{E a t^3}{4 b^3}$$

$$k_2 = \frac{AE}{l} = \frac{\pi d^2 E}{4 l}$$



From problem 5.1,

$$\omega_{1,2}^2 = \frac{k_1 + k_2}{2 m_1} + \frac{k_2}{2 m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$

$$= \left(\frac{E a t^3}{4 b^3} + \frac{\pi d^2 E}{4 l}\right)\frac{g}{2 W_1} + \frac{\pi d^2 E g}{8 l W_2}$$

$$\mp \sqrt{\left[\frac{1}{4}\left\{\left(\frac{E a t^3}{4 b^3} + \frac{\pi d^2 E}{4 l}\right)\frac{g}{W_1} + \frac{\pi d^2 E g}{4 l W_2}\right\}^2 - \frac{E^2 a t^3 \pi d^2 g^2}{16 l b^3 W_1 W_2}\right]}$$

5.21 From solution of Problem 5.1, we find that for $m_1 = m$, $m_2 = 2m$, $k_1 = k$ and $k_2 = 2k$,

$$\omega_1^2 = \left(2 - \sqrt{3}\right)\frac{k}{m}; \quad \omega_2^2 = \left(2 + \sqrt{3}\right)\frac{k}{m}$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{k_2}{-m_2 \omega_1^2 + k_2} = \frac{1}{-1 + \sqrt{3}}$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{k_2}{-m_2\,\omega_2^2 + k_2} = \frac{1}{-1-\sqrt{3}}$$

First mode shape:

$$\begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ \left(\frac{X_1^{(1)}}{\sqrt{3}-1}\right) \cos(\omega_1 t + \phi_1) \end{Bmatrix}$$

For the motion to be identical with the first normal mode, we need to have $X_1^{(2)} = 0$. This requires that (from Eq. (5.18)):

$$\frac{1}{r_2 - r_1}\left[\left\{-r_1\,x_1(0) + x_2(0)\right\}^2 + \frac{1}{\omega_2^2}\left\{r_1\,\dot{x}_1(0) - \dot{x}_2(0)\right\}^2\right]^{\frac{1}{2}} = 0$$

or

$$x_2(0) = r_1\,x_1(0) = \frac{x_1(0)}{\sqrt{3}-1}$$

$$\dot{x}_2(0) = r_1\,\dot{x}_1(0) = \frac{\dot{x}_1(0)}{\sqrt{3}-1}$$

5.22 Let $m_1 = m$, $m_2 = 2\,m$, $k_1 = k$, $k_2 = 2\,k$.
Initial conditions: $x_1(0) = 0$, $x_2(0) = 0.1$ m, $\dot{x}_1(0) = 0$, $\dot{x}_2(0) = 0$
Eqs. (5.18) yield:

$$X_1^{(1)} = \frac{1}{r_2 - r_1}\left[(0-0.1)^2\right]^{\frac{1}{2}} = \frac{0.1}{r_2 - r_1} = \frac{0.1}{\left(\frac{-1}{\sqrt{3}+1} - \frac{1}{\sqrt{3}-1}\right)} = -\frac{0.1}{\sqrt{3}}$$

$$X_1^{(2)} = \frac{1}{r_2 - r_1}\left[(0.1)^2\right]^{\frac{1}{2}} = \frac{0.1}{r_2 - r_1} = -\frac{0.1}{\sqrt{3}}$$

$$\phi_1 = \tan^{-1}(0) = 0$$
$$\phi_2 = \tan^{-1}(0) = 0$$

where ω_1 and ω_2 are given by Eq. (E3) of solution of Problem 5.1.

Resulting motion:

$$x_1(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2) = -\frac{0.1}{\sqrt{3}}\left\{\cos\omega_1 t + \cos\omega_2 t\right\}$$

$$x_2(t) = r_1\,X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2\,X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$= \left(\frac{1}{\sqrt{3}-1}\right)\left(-\frac{0.1}{\sqrt{3}}\right)\cos\omega_1 t + \left(-\frac{1}{\sqrt{3}+1}\right)\left(-\frac{0.1}{\sqrt{3}}\right)\cos\omega_2 t$$

$$= -\frac{0.1}{\sqrt{3}}\left[\left(\frac{1}{\sqrt{3}-1}\right)\cos\omega_1 t - \left(\frac{1}{\sqrt{3}+1}\right)\cos\omega_2 t\right]$$

5.23 $\omega_1^2 \ge (2\pi \times 10)^2 = 3947.8602 \ (rad/sec)^2$

From solution of problem 5.20, this inequality can be expressed as:

$$\omega_1^2 = \left(\frac{E a t^3}{4 b^3} + \frac{\pi d^2 E}{4 l}\right)\frac{g}{2 W_1} + \frac{\pi d^2 E g}{8 l W_2}$$

$$- \sqrt{\left[\frac{1}{4}\left\{\left(\frac{E a t^3}{4 b^3} + \frac{\pi d^2 E}{4 l}\right)\frac{g}{W_1} + \frac{\pi d^2 E g}{4 l W_2}\right\}^2 - \frac{E^2 a t^3 \pi d^2 g^2}{16 l b^3 W_1 W_2}\right]}$$

$$\ge 3947.8602 \qquad (E_1)$$

Data: $E = 30 \times 10^6$ psi, $W_1 = 1000$ lb, $W_2 = 500$ lb, $g = 386.4$ in/s²,

$b = 30$ in, $l = 60$ in.

Unknowns: a, t, d.

Let $a = 10\,t$ and $d = t$.

For this data, t is incremented from 0.1 in in increments of 0.01 in and the left hand side of the inequality (E_1) is evaluated. This gives a value of $t = 1.54$ in for satisfying (E_1).

$\therefore$ Design is $t = 1.54''$, $d = t = 1.54''$, $a = 10\,t = 15.4''$

5.24 With $k_{t1} = k_t$, $k_{t2} = 2\,k_t$, $J_1 = J_0$, $J_2 = 2 J_0$, $k_{t3} = 0$ and $M_{t1} = M_{t2} = 0$, Eqs. (5.20) give

$$J_0 \ddot{\theta}_1 + 3 k_t \theta_1 - 2 k_t \theta_2 = 0$$

$$2 J_0 \ddot{\theta}_2 - 2 k_t \theta_1 + 2 k_t \theta_2 = 0$$

For harmonic solution, $\theta_i(t) = \Theta_i \cos(\omega t + \phi)$; $i = 1, 2$,

$$\begin{bmatrix} (-\omega^2 J_0 + 3 k_t) & -2 k_t \\ -2 k_t & (-2\omega^2 J_0 + 2 k_t) \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

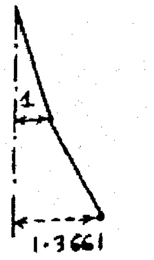
Frequency equation is

$$\begin{vmatrix} -\omega^2 J_0 + 3 k_t & -2 k_t \\ -2 k_t & -2\omega^2 J_0 + 2 k_t \end{vmatrix} = 2 J_0^2 \omega^4 - 8 J_0 k_t \omega^2 + 2 k_t^2 = 0$$

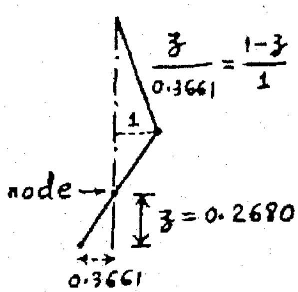
$$\omega^2 = (2 \mp \sqrt{3})\frac{k_t}{J_0} \ ; \quad \omega_1 = 0.5176\sqrt{\frac{k_t}{J_0}}, \quad \omega_2 = 1.9319\sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-J_0\,\omega_1^2 + 3\,k_t}{2\,k_t} = 1.3661$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-J_0\,\omega_2^2 + 3\,k_t}{2\,k_t} = -0.3661$$



First mode



Second mode

(5.25) Equation of motion of mass m: $\quad m\ddot{x} = -k_2(x - r\theta) \quad$ ---(E_1)

Equation of motion of cylinder of mass m_0 and mass moment of inertia $J_0 = \frac{1}{2} m_0 r^2$: $\quad J_0\ddot{\theta} = -k_1 r^2\theta - k_2(r\theta - x)r \quad$ ---(E_2)

For $x(t) = X\cos(\omega t + \phi)$ and $\theta(t) = \Theta\cos(\omega t + \phi)$, Eqs. ($E_1$) and ($E_2$) give the frequency equation

$$\begin{vmatrix} -m\omega^2 + k_2 & -k_2 r \\ -k_2 r & -\frac{1}{2} m_0 r^2\omega^2 + k_1 r^2 + k_2 r^2 \end{vmatrix} = 0$$

i.e. $\quad \omega^4 - \omega^2\left(\frac{k_2}{m} + \frac{2\{k_1 + k_2\}}{m_0}\right) + \frac{2\,k_1 k_2}{m_0\, m} = 0$

$$\omega_1^2,\ \omega_2^2 = \frac{k_2}{2m} + \frac{(k_1 + k_2)}{m_0} \mp \sqrt{\frac{1}{4}\left(\frac{k_2}{m} + \frac{2\,k_1}{m_0} + \frac{2\,k_2}{m_0}\right)^2 - \frac{2\,k_1 k_2}{m\,m_0}}$$

(5.26) For $J_1 = J_0$, $J_2 = 2J_0$, $k_{t_1} = k_{t_2} = k_{t_3} = k_t$, and $M_{t1} = M_{t2} = 0$, Eqs. (5.20) give

$$J_0\ddot{\theta}_1 + 2k_t\theta_1 - k_t\theta_2 = 0$$

$$2J_0\ddot{\theta}_2 - k_t\theta_1 + 2k_t\theta_2 = 0$$

For harmonic motion, these equations give

$$\begin{bmatrix} -\omega^2 J_0 + 2k_t & -k_t \\ -k_t & -2\omega^2 J_0 + 2k_t \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

from which the frequency equation can be obtained as

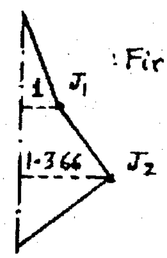
$$2J_0^2\omega^4 - 6J_0 k_t\,\omega^2 + 3\,k_t^2 = 0$$

$$\omega_1^2,\ \omega_2^2 = \frac{1}{2}(3 \mp \sqrt{3})\,\frac{k_t}{J_0}$$

$$\therefore\ \omega_1 = 0.79623\sqrt{\frac{k_t}{J_0}}\ ; \qquad \omega_2 = 1.53819\sqrt{\frac{k_t}{J_0}}$$

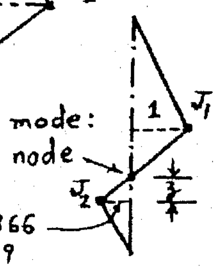
$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2 J_0 + 2k_t}{k_t} = 1.36603$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2 J_0 + 2k_t}{k_t} = -0.36603$$



: First mode

Second mode:



(5.27) Eqs. (5.20) give

$$J_0 \ddot{\theta}_1 + 6 k_t \theta_1 - 5 k_t \theta_2 = 0$$

$$5 J_0 \ddot{\theta}_2 - 5 k_t \theta_1 + 5 k_t \theta_2 = 0$$

These equations can be expressed as, for harmonic motion,

$$\begin{bmatrix} -\omega^2 J_0 + 6 k_t & -5 k_t \\ -5 k_t & -5\omega^2 J_0 + 5 k_t \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$J_0^2 \omega^4 - 7 k_t J_0 \omega^2 + k_t^2 = 0$$

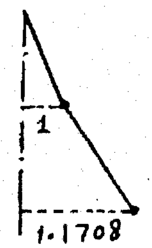
$$\omega_1^2, \omega_2^2 = \frac{k_t}{J_0}\left(\frac{7}{2} \mp \frac{1}{2}\sqrt{45}\right) = 0.1459 \frac{k_t}{J_0}, \; 6.8541 \frac{k_t}{J_0}$$

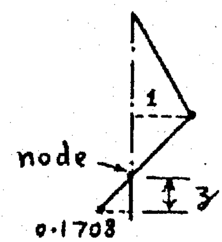
$$\omega_1 = 0.38197 \sqrt{\frac{k_t}{J_0}}, \quad \omega_2 = 2.61803 \sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2 J_0 + 6 k_t}{5 k_t} = 1.1708$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2 J_0 + 6 k_t}{5 k_t} = -0.1708$$

First mode



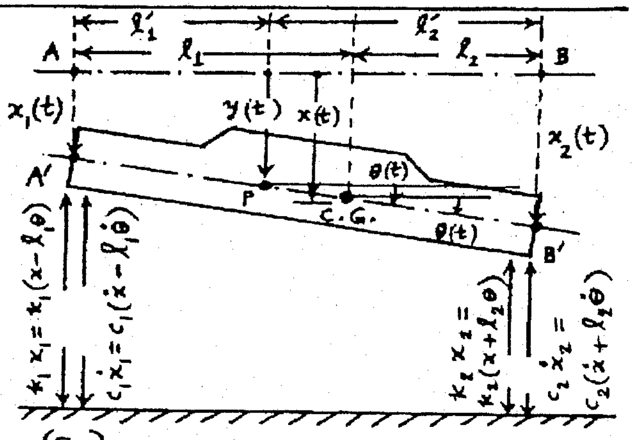


Second mode

$$\frac{z}{0.1708} = \frac{1-z}{1}$$

$$z = 0.1459$$

(5.28) (i) Using $x(t)$ and $\theta(t)$:



For translatory motion:

$$m\ddot{x} = -k_1(x - l_1\theta) - c_1(\dot{x} - l_1\dot{\theta}) - k_2(x + l_2\theta) - c_2(\dot{x} + l_2\dot{\theta}) \quad \text{---} \; (E_1)$$

For rotational motion about C.G.:

$$J_0\ddot{\theta} = k_1(x - l_1\theta) l_1 + c_1(\dot{x} - l_1\dot{\theta}) l_1 - k_2(x + l_2\theta) l_2 - c_2(\dot{x} + l_2\dot{\theta}) l_2 \quad \text{---} \; (E_2)$$

Eqs. (E_1) and (E_2) can be rewritten as

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_1 l_1 + c_2 l_2 \\ -c_1 l_1 + c_2 l_2 & c_1 l_1^2 + c_2 l_2^2 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_1 l_1 + k_2 l_2 \\ -k_1 l_1 + k_2 l_2 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(ii) Using $y(t)$ and $\theta(t)$:

For translatory motion:

$$m\ddot{y} = -k_1(y - l_1'\theta) - c_1(\dot{y} - l_1'\dot{\theta}) - k_2(y + l_2'\theta) - c_2(\dot{y} + l_2'\dot{\theta}) - me\ddot{\theta} \quad \text{---- } (E_3)$$

For rotational motion:

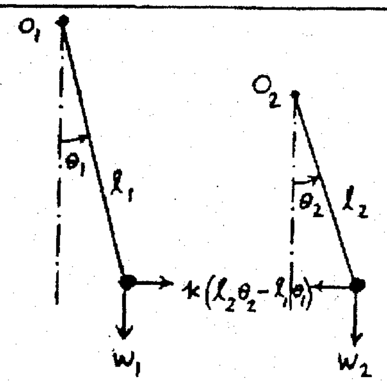
$$J_P\ddot{\theta} = k_1(y - l_1'\theta)l_1' + c_1(\dot{y} - l_1'\dot{\theta})l_1' - k_2(y + l_2'\theta)l_2' - c_2(\dot{y} + l_2'\dot{\theta})l_2' - me\ddot{y} \quad \text{---- } (E_4)$$

Eqs. (E_3) and (E_4) can be rewritten as

$$\begin{bmatrix} m & me \\ me & J_P \end{bmatrix} \begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_1 l_1' + c_2 l_2' \\ -c_1 l_1' + c_2 l_2' & c_1 l_1'^2 + c_2 l_2'^2 \end{bmatrix} \begin{Bmatrix} \dot{y} \\ \dot{\theta} \end{Bmatrix}$$

$$+ \begin{bmatrix} k_1 + k_2 & -k_1 l_1' + k_2 l_2' \\ -k_1 l_1' + k_2 l_2' & k_1 l_1'^2 + k_2 l_2'^2 \end{bmatrix} \begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

5.29 For small angular deflections, equations of motion are



$$m_1 l_1^2 \ddot{\theta}_1 = -W_1 l_1 \sin\theta_1 + k(l_2\theta_2 - l_1\theta_1) l_1 \cos\theta_1$$

$$m_2 l_2^2 \ddot{\theta}_2 = -W_2 l_2 \sin\theta_2 - k(l_2\theta_2 - l_1\theta_1) l_2 \cos\theta_2$$

or

$$m_1 l_1^2 \ddot{\theta}_1 + \theta_1(W_1 l_1 + k l_1^2) - k l_1 l_2 \theta_2 = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + \theta_2(W_2 l_2 + k l_2^2) - k l_1 l_2 \theta_1 = 0$$

For harmonic motion, $\theta_i(t) = \Theta_i \cos(\omega t + \phi)$; $i = 1,2$, we get

$$\begin{bmatrix} -\omega^2 m_1 l_1^2 + W_1 l_1 + k l_1^2 & -k l_1 l_2 \\ -k l_1 l_2 & -\omega^2 m_2 l_2^2 + W_2 l_2 + k l_2^2 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$\omega^4(m_1 m_2 l_1^2 l_2^2) - \omega^2 \left[m_2 l_2^2 (W_1 l_1 + k l_1^2) + m_1 l_1^2 (W_2 l_2 + k l_2^2)\right]$$
$$+ \left[W_1 l_1 W_2 l_2 + W_2 l_2 k l_1^2 + W_1 l_1 k l_2^2\right] = 0 \quad \text{---- } (E_1)$$

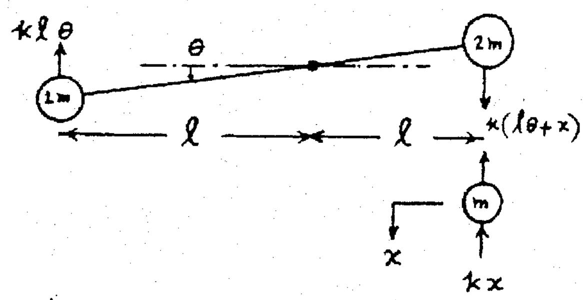
Roots of this equation give the natural frequencies ω_1 and ω_2.

Amplitude ratios are given by

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2 m_1 l_1^2 + W_1 l_1 + k l_1^2}{k l_1 l_2}$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2 m_1 l_1^2 + W_1 l_1 + k l_1^2}{k l_1 l_2} \qquad \text{where } W_1 = m_1 g,\ W_2 = m_2 g.$$

(5.30) Equations of motion:



$$4m\ell^2\ddot{\theta} = -k\ell\theta\cdot\ell - k(\ell\theta + x)\ell$$

$$m\ddot{x} = -kx - k(\ell\theta + x)$$

i.e. $$4m\ell^2\ddot{\theta} + 2k\ell^2\theta + k\ell x = 0$$

$$m\ddot{x} + 2kx + k\ell\theta = 0$$

For harmonic motion, these equations give

$$\begin{bmatrix} -4m\ell^2\omega^2 + 2k\ell^2 & k\ell \\ k\ell & -m\omega^2 + 2k \end{bmatrix} \begin{Bmatrix} \Theta \\ X \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is $\quad 4m^2\omega^4 - 10\,km\,\omega^2 + 3k^2 = 0$

$$\omega^2 = \frac{k}{m}\left(\frac{5}{4} \mp \frac{\sqrt{13}}{4}\right) = 0.3486\,\frac{k}{m}\,,\quad 2.1514\,\frac{k}{m}$$

$$\omega_1 = 0.5904\sqrt{\frac{k}{m}}\,,\qquad \omega_2 = 1.4668\sqrt{\frac{k}{m}}$$

Amplitude ratios are

$$r_1 = \frac{X^{(1)}}{\Theta^{(1)}} = \frac{-4m\ell^2\omega_1^2 + 2k\ell^2}{-k\ell} = -0.6056\,\ell$$

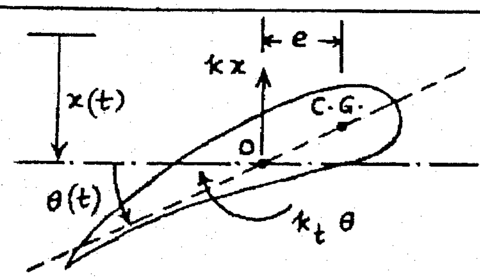
$$r_2 = \frac{X^{(2)}}{\Theta^{(2)}} = \frac{-4m\ell^2\omega_2^2 + 2k\ell^2}{-k\ell} = 6.6056\,\ell$$

Mode shapes are

$$\vec{X}^{(1)} = \begin{Bmatrix} \Theta^{(1)} \\ X^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.6056\ell \end{Bmatrix} \Theta^{(1)}$$

$$\vec{X}^{(2)} = \begin{Bmatrix} \Theta^{(2)} \\ X^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 6.6056\,\ell \end{Bmatrix} \Theta^{(2)}$$

(5.31) Equations of motion:



$$m(\ddot{x} - e\ddot{\theta}) = -kx$$

$$J_{CG}\,\ddot{\theta} = -k_t\theta - kxe$$

i.e. $$m\ddot{x} + kx - me\ddot{\theta} = 0$$

$$(J_0 - me^2)\ddot{\theta} + k_t\theta + kex = 0$$

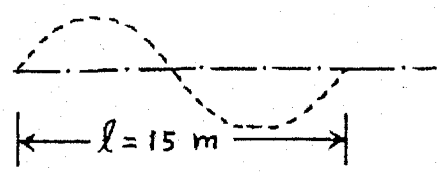
For harmonic motion, we get the frequency equation as

$$\begin{vmatrix} -m\omega^2 + k & me\omega^2 \\ ke & -(J_0 - me^2)\omega^2 + k_t \end{vmatrix} = 0$$

or $$(J_0 - me^2)m\,\omega^4 - (J_0\,k + m\,k_t)\,\omega^2 + k\,k_t = 0$$

Roots of this equation give the natural frequencies of the system.

(5.32) Speed becomes unfavorable when it is related to l as



$$v \, \tau_n = l$$

i.e., $$v = \frac{l}{\tau_n} = l \, f_n = \frac{l \, \omega_n}{2\pi}$$

Example 5.6 gives

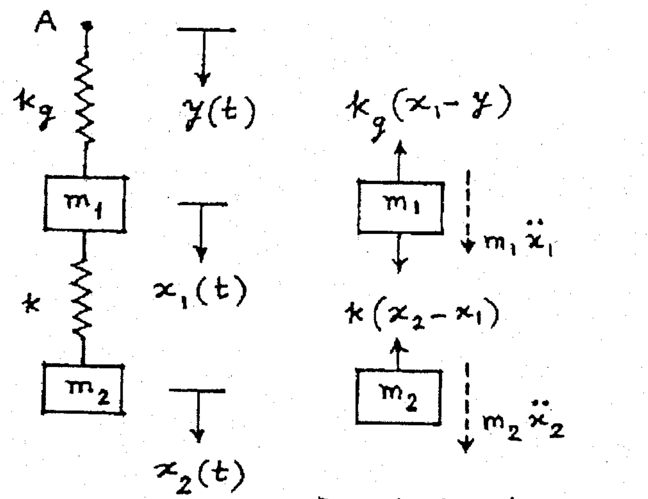
$\omega_1 = 5.8593$ rad/sec (bouncing)

$\omega_2 = 9.4341$ rad/sec (pitching)

$$\therefore \; v_1 = \frac{l \, \omega_1}{2\pi} = \frac{15 \, (5.8593)}{2\pi} = 13.9880 \text{ m/s} \quad \text{(bouncing)}$$

$$v_2 = \frac{l \, \omega_2}{2\pi} = \frac{15 \, (9.4341)}{2\pi} = 22.5222 \text{ m/s} \quad \text{(pitching)}$$

(5.33)



Free body diagrams of masses

Equations of motion:

$$m_1 \, \ddot{x}_1 + (k_g + k) \, x_1 - k \, x_2 = k_g \, y$$
$$m_2 \, \ddot{x}_2 - k \, x_1 + k \, x_2 = 0$$

Since velocity of crane in z-direction = 30 ft/min = 0.5 ft/sec, τ = time to complete one cycle = 10/0.5 = 20 sec, and $\omega = \frac{2\,\pi}{\tau} = \frac{2\,\pi}{20} = 0.31416$ rad/sec.

Base motion for m_1 (girder motion due to unevenness of rails):

$$y(t) = Y \sin \omega \, t$$

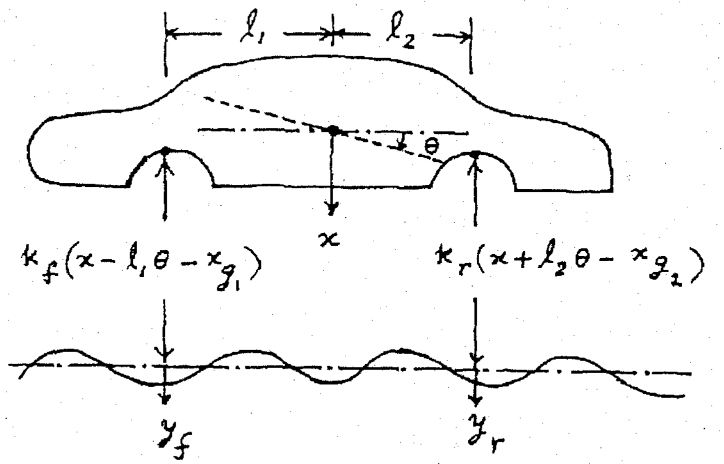
where Y = 2 in and $\omega = 0.31416$ rad/sec.

(5.34) Road surface varies sinusoidally with amplitude, Y = 0.05 m and wavelength, d = 10 m. If v = velocity of automobile (m/sec), time to travel one wave length = τ = d/v sec. τ = 10/v sec, $\omega = \frac{2\,\pi}{\tau} = \frac{2\,\pi\,v}{10}$ rad/sec.

v = 50 km/hr = $(50\,(10^3))/(60\,(60))$ = 13.8889 m/sec,

$J_0 = m \, r_g^2 = 1000 \, (0.9)^2 = 810$ kg–m^2.

Equations of motion:



y_f (y_r) = ground or base displacement of front (rear) wheels, downwards

For motion along x:

$$m\,\ddot{x} + x\,(k_f + k_r) + \theta\,(k_r\,\ell_2 - k_f\,\ell_1) = k_f\,y_f + k_r\,y_r \tag{1}$$

For motion along θ:

$$J_0\,\ddot{\theta} + x\,(\ell_2\,k_r - \ell_1\,k_f) + \theta\,(k_r\,\ell_2^2 + k_f\,\ell_1^2) = k_r\,\ell_2\,y_r - k_f\,\ell_1\,y_f \tag{2}$$

where the ground (base) motions can be expressed as

$$y_f(t) = Y\sin\omega\,t = 0.05\sin\frac{2\,\pi\,v}{10}\,t\ \ m \tag{3}$$

$$y_r(t) = Y\sin(\omega\,t - \phi) = 0.05\sin\left(\frac{2\,\pi\,v}{10}\,t - \frac{2\,\pi\,(\ell_1+\ell_2)}{d}\right)\ m \tag{4}$$

For given data, Eqs. (1) and (2) take the form:

$$1000\,\ddot{x} + 40\,(10^3)\,x + 15000\,\theta = 900\sin 8.7267\,t + 1100\sin(8.7267\,t - 1.5708) \tag{5}$$

$$810\,\ddot{\theta} + 15000\,x + 67500\,\theta = 1650\sin(8.7267\,t - 1.5708) - 900\sin 8.7267\,t \tag{6}$$

(5.35) Natural frequencies are given by:

$$\left[-\omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\right]\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1}$$

where m_1 = mass of pulley = $\frac{200}{386.4}$ lb–sec^2/in ; m_2 = mass of motor = $\frac{500}{386.4}$ lb–sec^2/in

X_1 = amplitude of pulley, X_2 = amplitude of motor,

$$\frac{E\,I}{\ell^3} = \frac{(30\,(10^6))\,(\frac{\pi}{64}\,(2^4))}{(90^3)} = 32.3210\ lb/in$$

Frequency equation becomes:

$$\begin{vmatrix} (-\omega^2 m_1 + k_{11}) & k_{12} \\ k_{12} & (-\omega^2 m_2 + k_{22}) \end{vmatrix} = 0$$

or

$$(m_1 m_2)\,\omega^4 - (k_{11}\, m_2 + k_{22}\, m_1)\,\omega^2 + (k_{11}\, k_{22} - k_{12}^2) = 0 \tag{2}$$

From known data, Eq. (2) can be expressed as:

$$0.6698\,\omega^4 - 11563.2894\,\omega^2 + 7.3108\,(10^6) = 0 \tag{3}$$

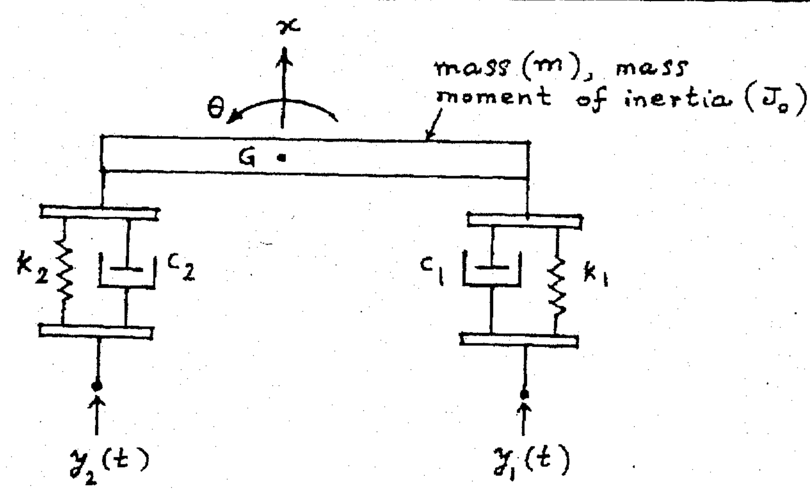
Roots of Eq. (3):

$$\omega^2 = 657.26642,\quad 16606.5300 \tag{4}$$

or

$$\omega_1 = 25.6372 \text{ rad/sec},\quad \omega_2 = 128.8663 \text{ rad/sec}$$

5.36



1. Model the bicycle and the rider as a two d.o.f system as shown in figure.
2. Find the equivalent stiffness (k_1) and damping coefficient (c_1) of the front wheel in the vertical direction.
3. Find the equivalent stiffness (k_2) and damping coefficient (c_2), if applicable, of the rear wheel in the vertical direction.
4. Describe the road roughness under the wheels as $y_1(t)$ and $y_2(t)$.
5. Derive the equations of motion of the system subjected to base excitation.
6. Solve the resulting system of equations to find the steady state response.

5.37 (a)

Choose unknown coordinates as x(t) and $\theta(t)$. Equations of motion:

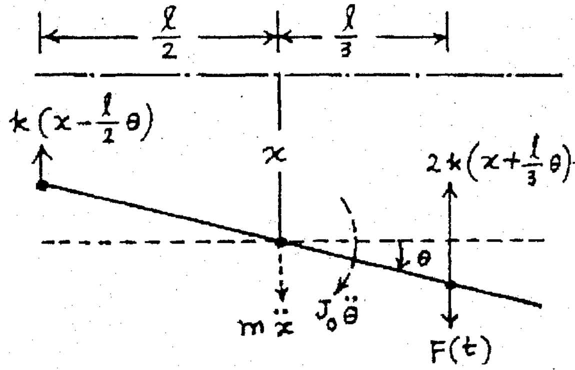
$$m\,\ddot{x} = -k\,(x - \ell\,\theta/2) - 2\,k\,(x + \ell\,\theta/3) + F(t)$$

$$J_0\,\ddot{\theta} = k\,(x - \ell\,\theta/2)\,(\ell/2) - 2\,k\,(x + \ell\,\theta/3)\,(\ell/3) + F(t)\,(\ell/3)$$

or

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 3\,k & k\,\ell/6 \\ k\,\ell/6 & 17\,k\,\ell^2/36 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} F(t) \\ \ell\,F(t)/3 \end{Bmatrix}$$

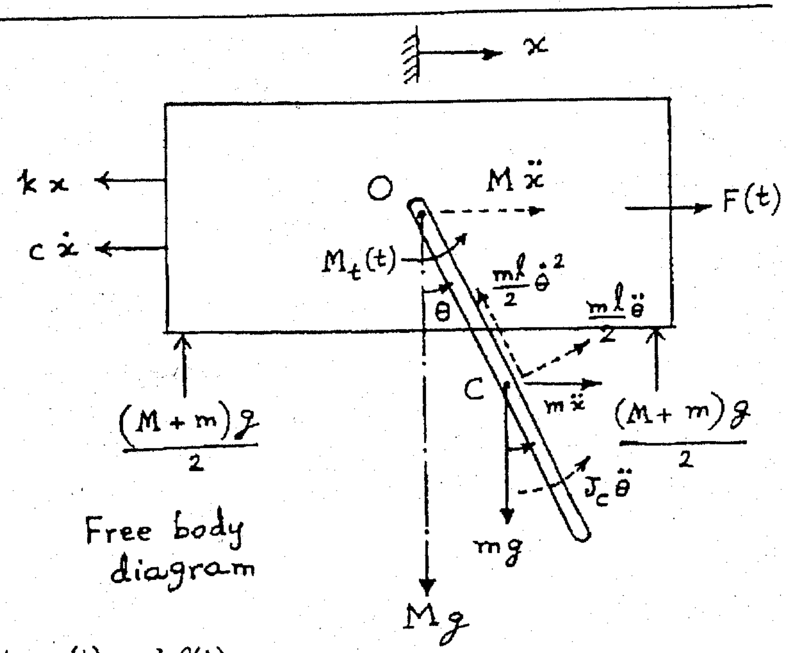
where $J_0 = \dfrac{m\,\ell^2}{12}$ and $F(t) = F_0 \sin\omega\,t$.



(b)

Elastic or static coupling.

5.38



Equations of motion with coordinates $x(t)$ and $\theta(t)$:

For motion along x:

$$M\,\ddot{x} = -\,k\,x - c\,\dot{x} - (m\,\ell/2)\,\ddot{\theta}\cos\theta - m\,\ddot{x} + (m\,\ell/2)\,\dot{\theta}^2 \sin\theta + F(t) \tag{1}$$

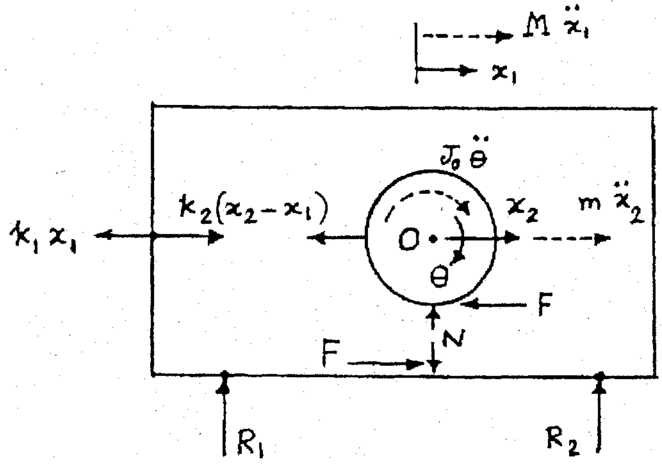
For rotation about O:

$$J_c\,\ddot{\theta} + (m\,\ell/2)\,\ddot{\theta}\,(\ell/2) + m\,\ddot{x}\,(\ell\cos\theta/2) = -\,m\,g\,(\ell/2)\sin\theta + M_t(t) \tag{2}$$

Using $J_c = \dfrac{1}{12}\,m\,\ell^2$, $\cos\theta \approx 1$ and $\sin\theta \approx \theta$ and neglecting the nonlinear term involving $\dot{\theta}^2$ in Eq. (1), Eqs. (1) and (2) can be rewritten in matrix form as:

$$\begin{bmatrix} (M+m) & m\,\ell/2 \\ m\,\ell/2 & (J_c + m\,\ell^2/4) \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & m\,g\,\ell/2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} F(t) \\ M_t(t) \end{Bmatrix}$$

(5.39)



Free body diagram

N = normal reaction between cylinder and trailer, F = friction force, R_1, R_2 = reactions between trailer and ground.

Equation of motion for linear motion of cylinder:

$$\sum F = m\ddot{x}_2 \quad \text{or} \quad m\ddot{x}_2 = -F - k_2(x_2 - x_1) \tag{1}$$

Equation of motion for rotational motion of cylinder:

$$\sum M_0 = J_0\ddot{\theta} \quad \text{or} \quad J_0\ddot{\theta} = F\,r \tag{2}$$

where $J_0 = \frac{1}{2}m\,r^2$ and $\theta = \frac{x_2 - x_1}{r}$.

Equation of motion for linear motion of trailer:

$$\sum F = M\ddot{x}_1 \quad \text{or} \quad M\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1) + F \tag{3}$$

Eq. (2) gives

$$F = \frac{J_0\ddot{\theta}}{r} = \frac{1}{r}\left(\frac{1}{2}m\,r^2\right)\left(\frac{\ddot{x}_2 - \ddot{x}_1}{r}\right) = \frac{m}{2}(\ddot{x}_2 - \ddot{x}_1) \tag{4}$$

Substitution of Eq. (4) into Eqs. (1) and (3) yields the equations of motion as:

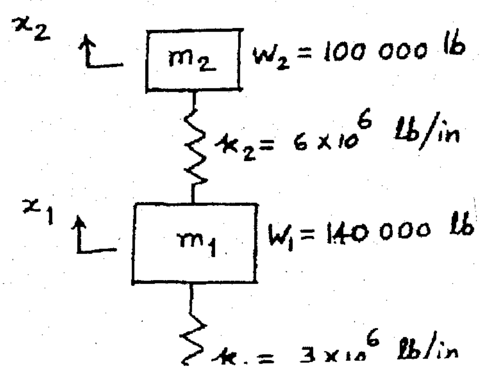
$$\frac{3m}{2}\ddot{x}_2 - \frac{1}{2}m\ddot{x}_1 - k_2x_1 + k_2x_2 = 0 \tag{5}$$

$$\left(M + \frac{m}{2}\right)\ddot{x}_1 - \frac{m}{2}\ddot{x}_2 + x_1(k_1 + k_2) - k_2x_2 = 0 \tag{6}$$

(5.40) (a) Natural frequencies of the system:

From problem 5.1,

$$\omega_{1,2}^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1k_2}{m_1m_2}}$$



$$\frac{\omega_{1,2}^2}{g} = \frac{9\times10^6}{28\times10^4} + \frac{6\times10^6}{20\times10^4}$$

$$\mp \sqrt{\frac{1}{4}\left(\frac{9\times10^6}{14\times10^4} + \frac{6\times10^6}{10\times10^4}\right)^2 - \frac{18\times10^{12}}{140\times10^8}}$$

or

$\omega_1 = 66.3408$ rad/sec

$\omega_2 = 208.8557$ rad/sec

(b) Initial conditions of the system:

Let v_2 = initial velocity of anvil and frame just after the impact of tup

From conservation of momentum principle,

momentum of tup plus momentum of anvil just before impact = momentum of tup plus momentum of anvil just after impact

i.e., $$m_{tup}\, v_{tup} + m_{anvil}\,(0) = m_{tup}\, v_0 + m_{anvil}\, v_2 \qquad (E_1)$$

where v_0 = velocity of rebound of tup after impact

Also,

$$\text{coefficient of restitution } (e) = \left(\frac{\text{relative velocity after impact}}{\text{relative velocity before impact}}\right)$$

i.e., $$e = \frac{v_2 - v_0}{v_{tup}} \quad \text{or} \quad v_0 = v_2 - e\, v_{tup} \qquad (E_2)$$

From Eqs. (E_1) and (E_2),

$$v_2 = \frac{m_{tup}\, v_{tup}\,(1+e)}{m_{tup} + m_{anvil}} \qquad (E_3)$$

For given data,

$$v_2 = \frac{5000\,(180)\,(1+0.5)}{105\,000} = 12.8571 \text{ in/sec}$$

$\therefore$ Initial conditions are:

$x_1(0) = 0, \quad \dot{x}_1(0) = 0$

$x_2(0) = 0, \quad \dot{x}_2(0) = 12.8571$ in/sec

(c) Displacements of anvil and foundation block:

We can use results of section 5.3 with $k_3 = 0$.

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\omega_1^2 + k_1 + k_2}{k_2} = \frac{-\frac{140\,000}{386.4}(4401.096) + 9\times10^6}{6\times10^6}$$

$$= 1.2342$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1\omega_2^2 + k_1 + k_2}{k_2} = \frac{-\frac{140\,000}{386.4}(43\,620.696) + 9\times10^6}{6\times10^6}$$

$$= -1.1341$$

Response of the system can be computed using Eqs. (5.18):

$$X_1^{(1)} = \frac{1}{r_2 - r_1}\left(\frac{\dot{x}_2}{\omega_1}\right) = \frac{1}{-2.3683}\left(\frac{12.8571}{66.3408}\right) = -0.08183 \text{ in}$$

$$X_1^{(2)} = \frac{1}{r_2 - r_1}\left(\frac{\dot{x}_2}{\omega_2}\right) = \frac{1}{-2.3683}\left(\frac{12.8571}{208.8557}\right) = -0.02599 \text{ in}$$

$$\phi_1 = \tan^{-1}\left\{\frac{\dot{x}_2(0)}{0}\right\} = \frac{\pi}{2}$$

$$\phi_2 = \tan^{-1}\left\{-\frac{\dot{x}_2(0)}{0}\right\} = -\frac{\pi}{2}$$

Response is given by Eqs. (5.15):

$$x_1(t) = X_1^{(1)}\cos(\omega_1 t + \phi_1) + X_1^{(2)}\cos(\omega_2 t + \phi_2)$$

$$= -0.08183\cos\left(66.3408\,t + \frac{\pi}{2}\right) - 0.02599\cos\left(208.8557\,t - \frac{\pi}{2}\right) \text{ in.}$$

$$x_2(t) = r_1 X_1^{(1)}\cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)}\cos(\omega_2 t + \phi_2)$$

$$= -0.1010\cos\left(66.3408\,t + \frac{\pi}{2}\right) + 0.02948\cos\left(208.8557\,t - \frac{\pi}{2}\right) \text{ in.}$$

5.41 (a) Natural frequencies:

Equations of motion:

$$m_1\ddot{x}_1 + k_1 x_1 - k_1 x_2 = F_1(t) \qquad (E_1)$$

$$m_2\ddot{x}_2 + (k_1 + k_2)x_2 - k_1 x_1 = 0 \qquad (E_2)$$

Frequency equation:

$$\begin{vmatrix} -\omega^2 m_1 + k_1 & -k_1 \\ -k_1 & -m_2\omega^2 + k_1 + k_2 \end{vmatrix} = 0$$

or
$$\omega^4 - \left(\frac{k_1}{m_1} + \frac{k_1+k_2}{m_2}\right)\omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0$$

$$\therefore\ \omega_{1,2}^2 = \frac{k}{2m_1} + \frac{k_1+k_2}{2m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1}{m_1} + \frac{k_1+k_2}{m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$

Here $m_1 = 2\times10^5$ kg, $m_2 = 2.5\times10^5$ kg, $k_1 = 150\times10^6$ N/m and $k_2 = 75\times10^6$ N/m.

$$\omega_{1,2}^2 = \frac{150\times10^6}{4\times10^5} + \frac{225\times10^6}{5\times10^5} \mp \sqrt{\frac{1}{4}\left(\frac{150\times10^6}{2\times10^5} + \frac{225\times10^6}{2.5\times10^5}\right)^2 - \frac{150\times75\times10^{12}}{5\times10^{10}}}$$

$$= 150,\ 1500\quad (\text{rad/sec})^2$$

$$\therefore\ \omega_1 = 12.2474 \text{ rad/sec},\quad \omega_2 = 38.7298 \text{ rad/sec}$$

(b) Response:

Assuming zero initial conditions, the Laplace transforms of (E_1) and (E_2) can be written as

$$m_1 s^2\, \bar{x}_1(s) + k_1\, \bar{x}_1(s) - k_1\, \bar{x}_2(s) = \bar{F}_1(s)$$
$$m_2 s^2\, \bar{x}_2(s) + (k_1+k_2)\, \bar{x}_2(s) - k_1\, \bar{x}_1(s) = 0$$

i.e.
$$(m_1 s^2 + k_1)\, \bar{x}_1(s) - k_1\, \bar{x}_2(s) = \bar{F}_1(s)$$
$$-k_1\, \bar{x}_1(s) + (m_2 s^2 + k_1 + k_2)\, \bar{x}_2(s) = 0$$

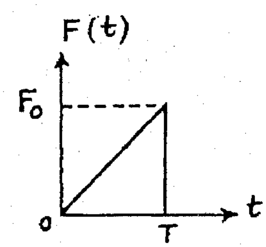
Solution of these equations gives

$$\bar{x}_1(s) = \left\{\frac{(m_2 s^2 + k_1 + k_2)}{m_1 m_2 s^4 + s^2(m_1 k_1 + m_1 k_2 + m_2 k_1) + k_1 k_2}\right\} \bar{F}_1(s) \qquad \text{----}(E_3)$$

$$\bar{x}_2(s) = \left\{\frac{k_1}{m_1 m_2 s^4 + s^2(m_1 k_1 + m_1 k_2 + m_2 k_1) + k_1 k_2}\right\} \bar{F}_1(s) \qquad \text{----}(E_4)$$

For the forcing function given,

$$\bar{F}_1(s) = \bar{F}(s) = \frac{F_0}{T}\left\{\frac{1}{s^2} - e^{-Ts}\left(\frac{T}{s} - \frac{1}{s^2}\right)\right\} \qquad \text{----}(E_5)$$



For the given data, Eqs. (E_3) to (E_5) become

$$\bar{x}_1(s) = \frac{2.5\times10^5\, s^2 + 225\times10^6}{(5\times10^{10}\, s^4 + 825\times10^{11}\, s^2 + 11250\times10^{12})}\ \bar{F}_1(s)$$

$$= \frac{s^2 + 900}{2\times10^5\, s^4 + 330\times10^6\, s^2 + 45\times10^9}\ \bar{F}_1(s) \qquad \text{----}(E_6)$$

$$\bar{x}_2(s) = \frac{150 \times 10^6}{5\times 10^{10}\, s^4 + 825 \times 10^{11}\, s^2 + 11250 \times 10^{12}} \; \bar{F}_1(s)$$

$$= \frac{30}{10^4\, s^4 + 165 \times 10^5\, s^2 + 225 \times 10^7} \; \bar{F}_1(s) \qquad \text{----} \; (E_7)$$

where $\bar{F}_1(s) = 2\times 10^5 \left[\frac{1}{s^2} - e^{-0.5 s} \left(\frac{1}{2s} - \frac{1}{s^2} \right) \right]$ ---- (E_8)

The inverse transforms of (E_6) and (E_7) yield $x_1(t)$ and $x_2(t)$.

5.42 Equations of motion for free vibration are (from Eqs. (5.1) and (5.2)):

$$\left.\begin{aligned} m_1 \ddot{x}_1 + (c_1+c_2)\dot{x}_1 + (k_1+k_2)x_1 - c_2\dot{x}_2 - k_2 x_2 &= 0 \\ m_2\ddot{x}_2 + (c_2+c_3)\dot{x}_2 + (k_2+k_3)x_2 - c_2\dot{x}_1 - k_2 x_1 &= 0 \end{aligned}\right\} \qquad (E_1)$$

Assuming the solution as

$$x_i(t) = \mathcal{X}_i\, e^{s_i t} \quad ; \quad i = 1,2$$

Eqs. (E_1) can be rewritten as

$$\begin{bmatrix} m_1 s^2 + (c_1+c_2)s + (k_1+k_2) & -(c_2 s + k_2) \\ -(c_2 s + k_2) & m_2 s^2 + (c_2+c_3)s + (k_2+k_3) \end{bmatrix} \begin{Bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (E_2)$$

For a non-trivial solution of Eqs. (E_2),

$$\begin{vmatrix} m_1 s^2 + (c_1+c_2)s + k_1+k_2 & -(c_2 s + k_2) \\ -(c_2 s + k_2) & m_2 s^2 + (c_2+c_3)s + k_2+k_3 \end{vmatrix} = 0$$

i.e.,

$$s^4(m_1 m_2) + s^3\left[m_1(c_2+c_3) + m_2(c_1+c_2)\right] + s^2\left[m_1(k_2+k_3) + (c_1+c_2)(c_2+c_3) + m_2(k_1+k_2) - c_2^2\right] + s\left[(c_1+c_2)(k_2+k_3) + (c_2+c_3)(k_1+k_2) - 2c_2k_2\right] + \left[(k_1+k_2)(k_2+k_3) - k_2^2\right] = 0 \qquad (E_3)$$

This equation can be expressed as

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0 \qquad (E_4)$$

where $a_0, a_1, \ldots, a_4$ can be identified by comparing Eqs. (E_4) and (E_3).

Nature of possible solutions:

If s_1, s_2, s_3 and s_4 are the roots of Eq. (E_4), the general solution of the system can be expressed as

$$\left.\begin{aligned} x_1(t) &= C_1^{(1)} e^{s_1 t} + C_1^{(2)} e^{s_2 t} + C_1^{(3)} e^{s_3 t} + C_1^{(4)} e^{s_4 t} \\ x_2(t) &= C_2^{(1)} e^{s_1 t} + C_2^{(2)} e^{s_2 t} + C_2^{(3)} e^{s_3 t} + C_2^{(4)} e^{s_4 t} \end{aligned}\right\} \qquad (E_5)$$

where the constants $C_1^{(i)}$, $i = 1$ to 4, can be found from the four initial conditions of the system, namely, $x_1(0)$, $x_2(0)$, $\dot{x}_1(0)$ and $\dot{x}_2(0)$. The ratios of amplitudes $C_1^{(i)}/C_2^{(i)}$ can be determined from Eqs. (E_2) as

$$\frac{C_1^{(i)}}{C_2^{(i)}} = \frac{c_2 s_i + k_2}{m_1 s_i^2 + (c_1+c_2) s_i + k_1 + k_2} = \frac{m_2 s_i^2 + (c_2+c_3) s_i + k_2 + k_3}{c_2 s_i + k_2};$$

$$i = 1, 2, 3, 4 \qquad (E_6)$$

If any root s_i has a positive real part, $x_1(t)$ and $x_2(t)$ will increase with time. If all s_i have negative real parts as

$$s_j = -r_j + i\,\omega_j$$

then the solution, $x_1(t)$, can be expressed as

$$x_1(t) = \sum_{j=1}^{4} C_1^{(j)} e^{-r_j t} e^{i \omega_j t} = \sum_{j=1}^{4} C_1^{(j)} e^{-r_j t} \left(\cos \omega_j t + i \sin \omega_j t\right)$$

If two roots s_1 and s_2 are complex conjugates as

$$s_1 = -(r_1 + i\,\omega_1) \quad \text{and} \quad s_2 = -(r_1 - i\,\omega_1),$$

$x_1(t)$ can be expressed as

$$\begin{aligned} x_1(t) = \; & e^{-r_1 t} \left\{ C_1^{(1)} \cos \omega_1 t - i\, C_1^{(1)} \sin \omega_1 t \right\} \\ & + e^{-r_1 t} \left\{ C_1^{(2)} \cos \omega_1 t + i\, C_1^{(2)} \sin \omega_1 t \right\} \\ & + C_1^{(3)} e^{s_3 t} + C_1^{(4)} e^{s_4 t} \end{aligned}$$

Similar expressions can be derived for $x_2(t)$.

(5.43) $m_1 = m_2 = 10$ kg, $k_1 = k_2 = 2000$ N/m, $k_3 = 2$ N/m,
$c_1 = 100$ N-s/m, $c_2 = c_3 = 1$ N-s/m.

Equations (E_3) and (E_4) in the solution of problem 5.42 give

$a_0 = m_1 m_2 = 100$

$a_1 = 10(2) + 10(101) = 1030$

$a_2 = 10(2002) + 101(2) + 10(4000) - 1 = 60221$

$a_3 = 101(2002) + 2(4000) - 2(1)(2000) = 206202$

$a_4 = 4000(2002) - 4 \times 10^6 = 4\,008\,000$

and

$$100\, s^4 + 1030\, s^3 + 60\,221\, s^2 + 206\,202\, s + 4\,008\,000 = 0 \qquad (E_1)$$

Using PROGRAM 10, the roots of Eq. (E_1) can be found as

$$s_{1,2} = -1.4714 \pm i\ 8.8272 \qquad (E_2)$$

$$s_{3,4} = -3.6786 \pm i\ 22.0668 \qquad (E_3)$$

Thus the solution is given by

$$x_1(t) = \sum_{j=1}^{4} C_1^{(j)}\, e^{s_j t}, \quad x_2(t) = \sum_{j=1}^{4} C_2^{(j)}\, e^{s_j t} \qquad (E_4, E_5)$$

where $C_1^{(j)}$, $j = 1,2,3,4$, can be found from the initial conditions, and the ratios of amplitudes $\left\{ \dfrac{C_1^{(j)}}{C_2^{(j)}} \right\}$ can be obtained from Eq. (E_6) in problem 5.42:

$$\frac{C_1^{(j)}}{C_2^{(j)}} = \frac{c_2\, s_j + k_2}{m_1\, s_j^2 + (c_1 + c_2)\, s_j + k_1 + k_2} = \frac{s_j + 2000}{10\, s_j^2 + 101\, s_j + 4000};$$

$$j = 1,2,3,4 \qquad (E_6)$$

For $j = 1$, $s_1 = -1.4714 + i\ 8.8272$ and (E_6) gives[†]

$$\alpha_1 = C_1^{(1)} / C_2^{(1)} = 0.6207 - i\ 0.1239 \qquad (E_7)$$

For $j = 2$, $s_2 = -1.4714 - i\ 8.8272$ and (E_6) gives

$$\alpha_2 = C_1^{(2)} / C_2^{(2)} = 0.6207 + i\ 0.1239 \qquad (E_8)$$

For $j = 3$, $s_3 = -3.6786 + i\ 22.0668$ and (E_6) yields

$$\alpha_3 = C_1^{(3)} / C_2^{(3)} = -1.3808 - i\ 0.7758 \qquad (E_9)$$

For $j = 4$, $s_4 = -3.6786 - i\,22.0668$ and (E_6) yields

$$\alpha_4 = C_1^{(4)} / C_2^{(4)} = -1.3808 + i\,0.7758 \qquad (E_{10})$$

Thus the solution of Eqs. (E_4) and (E_5) can be rewritten as

$$x_1(t) = \sum_{j=1}^{4} \alpha_j C_2^{(j)} e^{s_j t} \quad , \quad x_2(t) = \sum_{j=1}^{4} \alpha_j e^{s_j t} \qquad (E_{11}, E_{12})$$

Since the pairs (α_1, α_2), (α_3, α_4), (s_1, s_2) and (s_3, s_4) are complex conjugates, we can express them as

$$\left.\begin{aligned} \alpha_1, \alpha_2 &= p_1 \pm i\,q_1 \; ; \qquad \alpha_3, \alpha_4 = p_2 \pm i\,q_2 \\ s_1, s_2 &= u_1 \pm i\,v_1 \; ; \qquad s_3, s_4 = u_2 \pm i\,v_2 \end{aligned}\right\} \qquad (E_{13})$$

and (E_{11}) and (E_{12}) can be simplified further. However, we proceed directly with (E_{11}) and (E_{12}) and use the initial conditions to evaluate the constants $C_2^{(j)}$; $j = 1, 2, 3, 4$:

$$\left.\begin{aligned} x_1(0) &= \alpha_1 C_2^{(1)} + \alpha_2 C_2^{(2)} + \alpha_3 C_2^{(3)} + \alpha_4 C_2^{(4)} = 0.2 \\ x_2(0) &= C_2^{(1)} + C_2^{(2)} + C_2^{(3)} + C_2^{(4)} = 0.1 \\ \dot{x}_1(0) &= s_1 \alpha_1 C_2^{(1)} + s_2 \alpha_2 C_2^{(2)} + s_3 \alpha_3 C_2^{(3)} + s_4 s_4 C_2^{(4)} = 0 \\ \dot{x}_2(0) &= s_1 C_2^{(1)} + s_2 C_2^{(2)} + s_3 C_2^{(3)} + s_4 C_2^{(4)} = 0 \end{aligned}\right\} \qquad (E_{14})$$

Once $C_2^{(j)}$, $j = 1, 2, 3, 4$ are determined from Eqs. (E_{14}), the displacements of masses $x_1(t)$ and $x_2(t)$ can be obtained using Eqs. (E_{11}) and (E_{12}).

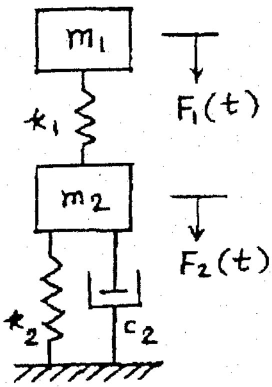
---

† If $s = a + ib$, $s^2 = (a^2 - b^2) + i\,(2ab)$

If $x = \dfrac{a + bi}{c + di}$, it can be rewritten as

$$x = \frac{(a+bi)(c-di)}{(c^2+d^2)} = \left(\frac{ac+bd}{c^2+d^2}\right) + i\left(\frac{bc-ad}{c^2+d^2}\right)$$

(5.44) $\omega = \dfrac{2\pi(1200)}{60} = 125.664$ rad/sec



$F_1(t) = m e \omega^2 \cos \omega t$

$= \left(\dfrac{0.5}{386.4}\right)(6)(125.664)^2 \cos 125.664 t$

$= 122.6044 \cos 125.664 t$ lb

$m_1 = 800/386.4 = 2.0704$ lb-s²/in

$m_2 = 2000/386.4 = 5.1760$ lb-s²/in

$k_1 = 2000$ lb/in, $k_2 = 1000$ lb/in, $c_2 = 200$ lb-s/in, $F_{10} = 122.6044$,

$F_{20} = 0$.

Equations of motion are [substitute $k_1 = k_2$, $c_1 = c_2$, $m_1 = m_2$, $F_1 = 0$, $k_2 = k_1$, $c_2 = 0$, $m_2 = m_1$, $F_2 = F_1$, $k_3 = 0$, $c_3 = 0$ in Eq. (5.3)]:

$$\begin{bmatrix} m_2 & 0 \\ 0 & m_1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_2 \\ \ddot{x}_1 \end{Bmatrix} + \begin{bmatrix} c_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{Bmatrix} + \begin{bmatrix} k_2 + k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} x_2 \\ x_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_1(t) \end{Bmatrix} \quad \ldots (E_1)$$

or

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix} \quad \ldots (E_2)$$

Comparing (E_2) with Eq. (5.27), we find that

$m_{11} = m_1$, $m_{12} = 0$, $m_{22} = m_2$, $c_{11} = 0$, $c_{12} = 0$, $c_{22} = c_2$, $k_{11} = k_1$, $k_{12} = 0$ and

$k_{22} = k_1 + k_2$.

Application of Eq. (5.31) leads to

$Z_{11}(i\omega) = -\omega^2 m_{11} + i\omega c_{11} + k_{11} = -m_1 \omega^2 + k_1$

$Z_{12}(i\omega) = -\omega^2 m_{12} + i\omega c_{12} + k_{12} = -k_1$

$Z_{22}(i\omega) = -\omega^2 m_{22} + i\omega c_{22} + k_{22} = -m_2 \omega^2 + i\omega c_2 + k_1 + k_2$

Response of the system can be expressed as

$x_j(t) = X_j e^{i\omega t} = X_j \cos \omega t$ (real part)

with X_j given by Eq. (5.35):

$$X_1(i\omega) = \frac{Z_{22}(i\omega) \cdot F_{10} - Z_{12}(i\omega) \cdot F_{20}}{Z_{11}(i\omega) \cdot Z_{22}(i\omega) - Z_{12}^2(i\omega)}$$

$$= \frac{(-m_2\omega^2 + i\omega c_2 + k_1 + k_2) F_{10}}{(-m_1\omega^2 + k_1)(-m_2\omega^2 + i\omega c_2 + k_1 + k_2) - k_1^2}$$

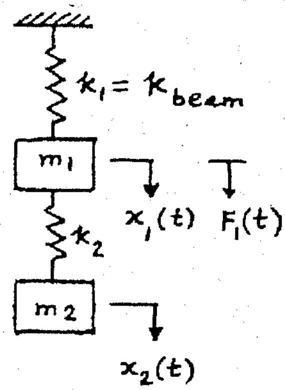
$$= \frac{\{-5.176\,(125.664)^2 + i(125.664)(200) + 3000\}\, 122.6044}{\{[-2.0704\,(125.664)^2 + 2000][-5.176\,(125.664)^2 + i(125.664)(200) + 3000] - 4\times 10^6\}}$$

$$= (-40.0042 - 0.01919\, i) \times 10^{-4} \text{ in}$$

$$X_2(i\omega) = \frac{-Z_{12}(i\omega)\, F_{10} + Z_{11}(i\omega)\, F_{20}}{Z_{11}(i\omega)\, Z_{22}(i\omega) - Z_{12}^2(i\omega)} = \frac{k\, F_{10}}{Z_{11}(i\omega)\, Z_{22}(i\omega) - Z_{12}^2(i\omega)}$$

$$= \frac{2000\,(122.6044)}{(24.1272 - 7.7143\, i)\, 10^8} = (0.9221 + 0.2948\, i) \times 10^{-4} \text{ in}$$

5.45 $k_1 = k_{beam} = \dfrac{192\, E\, (\frac{1}{12} a t^3)}{l^3} = \dfrac{16\, E\, a\, t^3}{l^3}$



Equations of motion:

$$\left.\begin{aligned} m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) &= F_1(t) = F_0 \cos\omega t \\ m_2 \ddot{x}_2 + k_2 (x_2 - x_1) &= 0 \end{aligned}\right\} \ldots (E_1)$$

Assuming harmonic response

$$x_j(t) = X_j \cos\omega t \quad ; \quad j = 1, 2$$

Eqs. (E_1) yield

$$X_1 = \frac{(k_2 - m_2\omega^2) F_0}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2}$$

$$X_2 = \frac{k_2\, F_0}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2}$$

For no steady state vibration of the beam, $X_1 = 0$ and hence the condition to be satisfied is

$$\frac{k_2}{m_2} = \omega^2$$

5.46 Equations of motion:

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = F_1(t) = F_0 \sin\omega t \quad \text{----} \; (E_1)$$

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0 \quad \text{----} \; (E_2)$$

We use $F_0\, e^{i\omega t}$ (with $i=\sqrt{-1}$) for $F_1(t)$ and consider only the imaginary part at the end.

Let $\quad x_j(t) = X_j\, e^{i\omega t} \quad ; \quad j = 1, 2$

Eqs. (E_1) and (E_2) become

$$-m_1\omega^2 X_1 e^{i\omega t} + (k_1+k_2) X_1 e^{i\omega t} - k_2 X_2 e^{i\omega t} = F_0 e^{i\omega t}$$

$$-m_2\omega^2 X_2 e^{i\omega t} + k_2 X_2 e^{i\omega t} - k_2 X_1 e^{i\omega t} = 0$$

i.e. $$[Z(i\omega)]\,\vec{X} = \vec{\underset{\sim}{F_0}} \qquad \text{----} \quad (E_3)$$

where, $\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$, $\vec{\underset{\sim}{F_0}} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$,

$$[Z(i\omega)] = \begin{bmatrix} Z_{11}(i\omega) & Z_{12}(i\omega) \\ Z_{21}(i\omega) & Z_{22}(i\omega) \end{bmatrix},$$

$Z_{11}(i\omega) = -m_1\omega^2 + k_1 + k_2, \quad Z_{12}(i\omega) = Z_{21}(i\omega) = -k_2,$

$Z_{22}(i\omega) = -m_2\omega^2 + k_2.$

Eqs. (5.35) give

$$X_1(i\omega) = \frac{(-m_2\omega^2 + k_2)F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2}$$

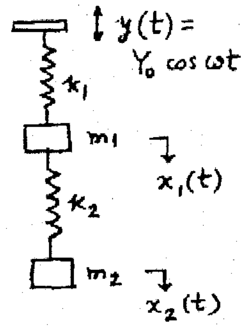
$$X_2(i\omega) = \frac{k_2 F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2}$$

Since $F_0 \sin\omega t = \mathrm{Im}\,(F_0 e^{i\omega t})$, $\quad x_j(t) = \mathrm{Im}\,(X_j\, e^{i\omega t}) = X_j \sin\omega t$

$$\therefore \quad x_1(t) = \frac{(-m_2\omega^2 + k_2)F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2}\, \sin\omega t$$

$$x_2(t) = \frac{k_2 F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2}\, \sin\omega t$$

5.47 Equations of motion:



$$m_1\ddot{x}_1 = -k_1(x_1 - y) - k_2(x_1 - x_2)$$

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1)$$

or $$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = k_1 Y_0 \cos\omega t \quad \text{---} (E_1)$$

$$m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0 \quad \text{---} (E_2)$$

As there is no damping, the masses vibrate either in phase or 180° out of phase with respect to the base motion. Hence the response can be taken as

$$x_j(t) = X_j \cos \omega t \quad ; \quad j = 1,2 \qquad \text{---} (E_3)$$

Eqs. (E_1) and (E_2) reduce to

$$(-\omega^2 m_1 + k_1 + k_2)\, X_1 - k_2\, X_2 = k_1 Y_0$$

$$-k_2\, X_1 + (-\omega^2 m_2 + k_2)\, X_2 = 0$$

i.e. $\left[Z(i\omega)\right] \vec{X} = \vec{F}$ where $\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$, $\vec{F} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} = \begin{Bmatrix} k_1 Y_0 \\ 0 \end{Bmatrix}$,

$$Z_{11}(i\omega) = -\omega^2 m_1 + k_1 + k_2, \qquad Z_{12}(i\omega) = Z_{21}(i\omega) = -k_2,$$

$$Z_{22}(i\omega) = -\omega^2 m_2 + k_2.$$

Eqs. (5.35) and (E_3) give

$$x_1(t) = \frac{(-\omega^2 m_2 + k_2)\, k_1 Y_0}{(-m_1 \omega^2 + k_1 + k_2)(-m_2 \omega^2 + k_2) - k_2^2} \cos \omega t$$

$$x_2(t) = \frac{k_1 k_2 Y_0}{(-m_1 \omega^2 + k_1 + k_2)(-m_2 \omega^2 + k_2) - k_2^2} \cos \omega t$$

5.48 Equations of motion:

$$m_1 \ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2) x_1 - c_2 \dot{x}_2 - k_2 x_2 = F_1(t) = F_0\, e^{i\omega t}$$

$$m_2 \ddot{x}_2 + c_2 \dot{x}_2 + k_2 x_2 - c_2 \dot{x}_1 - k_2 x_1 = 0$$

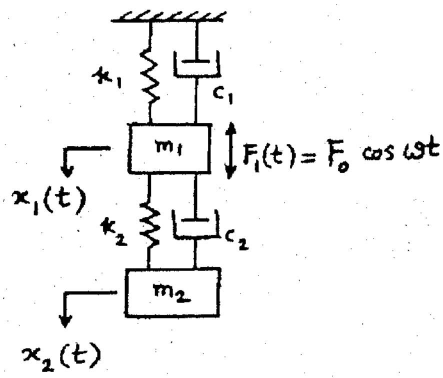
Let $x_j(t) = X_j\, e^{i\omega t}$; $j = 1,2$

$k_1 = k_2 = 500$ N/m

$m_1 = m_2 = 1$ kg

$c_1 = c_2 = c = 200$ N·s/m

$\omega = 1$ rad/s



Equations of motion become

$$\left[-\omega^2 m_1 + i\omega (c_1 + c_2) + k_1 + k_2\right] X_1 - (i\omega c_2 + k_2)\, X_2 = F_0 \qquad \text{---} (E_1)$$

$$-(i\omega c_2 + k_2)\, X_1 + \left[-\omega^2 m_2 + i\omega c_2 + k_2\right] X_2 = 0 \qquad \text{---} (E_2)$$

For given data, (E_1) and (E_2) become

$$\left[Z(i\omega)\right] \vec{X} = \vec{F}_0 \qquad \text{---} (E_3)$$

where $Z_{11}(i\omega) = 400\, i + 999$

$$Z_{12}(i\omega) = Z_{21}(i\omega) = -200\, i - 500$$

$$Z_{22}(i\omega) = 200\, i + 499$$

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}, \quad \vec{F}_0 = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

Solution of (E_3) is, using Eqs. (5.35),

$$X_1 = \frac{(200i+499)\,F_0}{(400i+999)(200i+499)-(-200i-500)^2} = \frac{(200i+499)\,F_0}{(199400i+208501)}$$

$$= \frac{(200i+499)(-199400i+208501)\,F_0}{(199400i+208501)(-199400i+208501)}$$

$$= (17.2915\times10^{-4} - 6.9444\times10^{-4}\,i)\,F_0 \qquad ---- (E_4)$$

$$X_2 = \frac{(200i+500)\,F_0}{(400i+999)(200i+499)-(-200i-500)^2} = \frac{(200i+500)\,F_0}{(199400i+208501)}$$

$$= \frac{(200i+500)(-199400i+208501)\,F_0}{(199400i+208501)(-199400i+208501)}$$

$$= (17.3165\times10^{-4} - 6.9684\times10^{-4}\,i)\,F_0 \qquad ---- (E_5)$$

Final solution is given by the real parts as

$$x_1(t) = \mathrm{Re}(X_1 e^{i\omega t}) = \mathrm{Re}(X_1\cos\omega t + i X_1 \sin\omega t)$$

$$= \mathrm{Re}\big[(17.2915\times10^{-4} - 6.9444\times10^{-4}\,i)\,F_0\cos\omega t$$

$$+ (17.2915\times10^{-4}\,i + 6.9444\times10^{-4})\,F_0\sin\omega t\big]$$

$$= 17.2915\times10^{-4}\,F_0\cos\omega t + 6.9444\times10^{-4}\,F_0\sin\omega t \qquad --- (E_6)$$

$$x_2(t) = \mathrm{Re}(X_2 e^{i\omega t}) = \mathrm{Re}(X_2\cos\omega t + i X_2\sin\omega t)$$

$$= \mathrm{Re}\big[(17.3165\times10^{-4} - 6.9684\times10^{-4}\,i)\,F_0\cos\omega t$$

$$+ (17.3165\times10^{-4}\,i + 6.9684\times10^{-4})\,F_0\sin\omega t\big]$$

$$= 17.3165\times10^{-4}\,F_0\cos\omega t + 6.9684\times10^{-4}\,F_0\sin\omega t \qquad --- (E_7)$$

5.49 Equations of motion:

$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = F_{10}\cos\omega t = \mathrm{Re}(F_{10}\,e^{i\omega t})$$

$$m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2x_1 = F_{20}\cos\omega t = \mathrm{Re}(F_{20}\,e^{i\omega t})$$

Assuming $x_j(t) = X_j\,e^{i\omega t}$; $j = 1,2$ along with $F_j(t) = F_{j0}\,e^{i\omega t}$; $j=1,2$, the equations of motion can be expressed as

$$(-\omega^2 m_1 + k_1 + k_2)\,X_1 - k_2 X_2 = F_{10}$$

$$-k_2 X_1 + (-\omega^2 m_2 + k_2 + k_3)\,X_2 = F_{20}$$

i.e. $$[Z(i\omega)]\,\vec{X} = \vec{F_0} \qquad ---- (E_1)$$

where $Z_{11}(i\omega) = -\omega^2 m_1 + k_1 + k_2$, $Z_{12}(i\omega) = Z_{21}(i\omega) = -k_2$,

$Z_{22}(i\omega) = -\omega^2 m_2 + k_2 + k_3$,

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}, \quad \vec{F_0} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$$

Solution of (E_1) can be expressed, using Eqs. (5.35), as

$$X_1 = \frac{(-\omega^2 m_2 + k_2 + k_3) F_{10} + k_2 F_{20}}{(-\omega^2 m_1 + k_1 + k_2)(-\omega^2 m_2 + k_2 + k_3) - k_2^2} \quad ---- (E_2)$$

$$X_2 = \frac{k_2 F_{10} + (-\omega^2 m_1 + k_1 + k_2) F_{20}}{(-\omega^2 m_1 + k_1 + k_2)(-\omega^2 m_2 + k_2 + k_3) - k_2^2} \quad ---- (E_3)$$

Since X_1 and X_2 are real (since there is no damping), the final solution is given by

$$x_1(t) = X_1 \cos \omega t$$

$$x_2(t) = X_2 \cos \omega t$$

where X_1 and X_2 are given by (E_2) and (E_3).

5.50 From the solution of problem 5.46, we have

$$x_1(t) = \frac{(-m_2 \omega^2 + k_2) F_0}{(-m_1 \omega^2 + k_1 + k_2)(-m_2 \omega^2 + k_2) - k_2^2} \sin \omega t$$

$$x_2(t) = \frac{k_2 F_0}{(-m_1 \omega^2 + k_1 + k_2)(-m_2 \omega^2 + k_2) - k_2^2} \sin \omega t$$

For the data $F_1(t) = 50 \sin 4\pi t$, $F_0 = 50$ N, $\omega = 4\pi$ rad/s, $m_1 = 10$ kg, $m_2 = 5$ kg, $k_1 = 8000$ N/m and $k_2 = 2000$ N/m,

$$x_1(t) = \frac{(-5 \times 16 \pi^2 + 2000) 50}{(-10 \times 16 \pi^2 + 8000 + 2000)(-5 \times 16 \pi^2 + 2000) - (2000)^2} \sin 4\pi t$$

$$= 0.009773 \sin 4\pi t \text{ meters}$$

$$x_2(t) = \frac{2000(50)}{(-10 \times 16 \pi^2 + 8000 + 2000)(-5 \times 16 \pi^2 + 2000) - (2000)^2} \sin 4\pi t$$

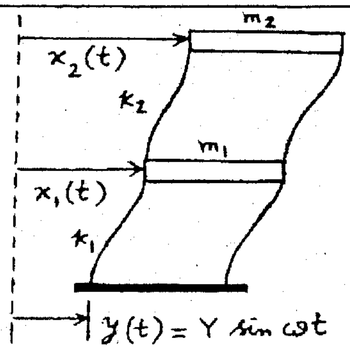
$$= 0.016148 \sin 4\pi t \text{ meters}$$

5.51 k_1 = total stiffness = 800 N/m

k_2 = total stiffness = 600 N/m

$m_1 = 50$ kg, $m_2 = 50$ kg

$Y = 0.2$ m, $\omega = \pi$ rad/s



Equations of motion:

$$m_1 \ddot{x}_1 = -k_1 (x_1 - y) - k_2 (x_1 - x_2)$$

$$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1)$$

i.e. $$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = k_1 y = k_1 Y \sin \omega t$$

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0$$

Assuming $x_i(t) = X_i \sin \omega t$; $i = 1, 2$, we get

$$(-m_1\omega^2 + k_1 + k_2) X_1 - k_2 X_2 = k_1 Y$$

$$-k_2 X_1 + (-m_2\omega^2 + k_2) X_2 = 0$$

For given data, these equations take the form

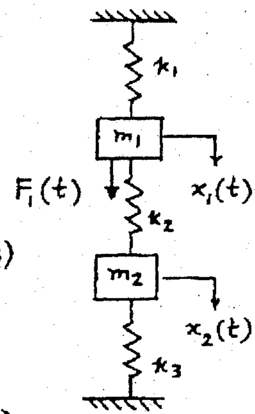
$$(-50\pi^2 + 1400) X_1 - 600 X_2 = (800)(0.2)$$

$$-600 X_1 + (-50\pi^2 + 600) X_2 = 0$$

Solution of these equations gives $X_1 = -0.06469$ m, $X_2 = -0.36439$ m

$\therefore$ $x_1(t) = -0.06469 \sin \pi t$ m ; $x_2(t) = -0.36439 \sin \pi t$ m.

5.52 Equations of motion:



$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2 x_2 = F_1(t) \quad \text{--- } (E_1)$$

$$m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2 x_1 = 0 \quad \text{--- } (E_2)$$

Laplace transforms of (E_1) and (E_2) are

$$m_1[s^2\bar{x}_1(s) - s\,x_1(0) - \dot{x}_1(0)] + (k_1+k_2)\bar{x}_1(s) - k_2\bar{x}_2(s) = \bar{F}_1(s)$$

$$m_2[s^2\bar{x}_2(s) - s\,x_2(0) - \dot{x}_2(0)] + (k_2+k_3)\bar{x}_2(s) - k_2\bar{x}_1(s) = 0$$

Rearranging these equations, we get

$$(m_1 s^2 + k_1 + k_2)\bar{x}_1(s) - k_2\bar{x}_2(s) = \bar{F}_1(s) + s m_1 x_1(0) + m_1\dot{x}_1(0) \quad \text{---- } (E_3)$$

$$-k_2\bar{x}_1(s) + (m_2 s^2 + k_2 + k_3)\bar{x}_2(s) = s m_2 x_2(0) + m_2\dot{x}_2(0) \quad \text{--- } (E_4)$$

When $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$, (E_3) and (E_4) give

$$(m s^2 + 2k)\bar{x}_1(s) - k\bar{x}_2(s) = \bar{F}_1(s) + s m x_1(0) + m\dot{x}_1(0) \quad \text{--- } (E_5)$$

$$-k\bar{x}_1(s) + (m s^2 + 2k)\bar{x}_2(s) = s m x_2(0) + m\dot{x}_2(0) \quad \text{--- } (E_6)$$

Solution of Eqs. (E_5) and (E_6) gives

$$\bar{x}_1(s) = \frac{(ms^2+2k)\{\bar{F}_1(s) + m\,x_1(0)\cdot s + m\,\dot{x}_1(0)\} + k\{m\,x_2(0)\cdot s + m\,\dot{x}_2(0)\}}{(ms^2+2k)(ms^2+2k) - k^2}$$

$$\bar{x}_2(s) = \frac{k\{\bar{F}_1(s) + m\,x_1(0)\cdot s + \dot{x}_1(0)\,m\} + (ms^2+2k)\{m\,x_2(0)\cdot s + m\,\dot{x}_2(0)\}}{(ms^2+2k)(ms^2+2k) - k^2}$$

These equations become, for $x_1(0) = x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$,
$F_1(t) = 5\,u(t)$, $\bar{F}_1(s) = \frac{5}{s}$, $m = 1$ and $k = 100$,

$$\bar{x}_1(s) = \frac{5(s^2+200)}{s\,[(s^2+200)^2 - 10000]} = \frac{5(s^2+200)}{s(s^2+300)(s^2+100)} \quad \text{--- } (E_7)$$

$$\bar{x}_2(s) = 100\left(\frac{5}{s}\right)\frac{1}{[(s^2+200)^2 - 10000]} = \frac{500}{s(s^2+300)(s^2+100)} \quad \text{--- } (E_8)$$

By expressing

$$\bar{x}_1(s) = \frac{5(s^2+200)}{s(s^2+300)(s^2+100)} = \frac{A_1}{s} + \frac{A_2 s + A_3}{s^2+100} + \frac{A_4 s + A_5}{s^2+300}$$

$$\bar{x}_2(s) = \frac{500}{s(s^2+300)(s^2+100)} = \frac{B_1}{s} + \frac{B_2 s + B_3}{s^2+100} + \frac{B_4 s + B_5}{s^2+300}$$

we can find $A_1, A_2, \ldots, B_1, B_2, \ldots$ (partial fractions method). This leads to

$$\bar{x}_1(s) = \frac{1}{30 s} - \frac{s}{40(s^2+100)} - \frac{s}{120(s^2+300)} \quad \text{---}(E_9)$$

$$\bar{x}_2(s) = \frac{1}{60 s} - \frac{s}{40(s^2+100)} + \frac{s}{120(s^2+300)} \quad \text{---}(E_{10})$$

Inverse Laplace transforms of (E_9) and (E_{10}) give

$$x_1(t) = \left(\frac{1}{30} - \frac{1}{40}\cos 10t - \frac{1}{120}\cos 10\sqrt{3}\,t\right) u(t)$$

$$x_2(t) = \left(\frac{1}{60} - \frac{1}{40}\cos 10t + \frac{1}{120}\cos 10\sqrt{3}\,t\right) u(t)$$

It is to be noted that $x_1(t) = \frac{1}{30}$ meter and $x_2(t) = \frac{1}{60}$ meter are the static deflections associated with 5 N static force applied to mass m_1. $\omega_1 = 10$ rad/s and $\omega_2 = 10\sqrt{3}$ rad/s are the two resonant frequencies associated with the two degree of freedom system.

(5.53) Equivalent mass of cylinder with respect to $x_2 = (m_2)_{eq} = m_2 + \frac{J_0}{r^2}$

Equations of motion: $m_1 \ddot{x}_1 = -k(x_1 - x_2)$

$$(m_2)_{eq}\, \ddot{x}_2 = -k(x_2 - x_1)$$

i.e.

$$m_1 \ddot{x}_1 + k x_1 - k x_2 = 0 \quad \text{----}(E_1)$$

$$\left(m_2 + \frac{J_0}{r^2}\right)\ddot{x}_2 + k x_2 - k x_1 = 0 \quad \text{----}(E_2)$$

Assuming $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1, 2$, Eqs. (E_1) and (E_2) can be written as

$$(-m_1\omega^2 + k)X_1 - kX_2 = 0$$

$$-kX_1 + \left(-\omega^2\left\{m_2 + \frac{J_0}{r^2}\right\} + k\right)X_2 = 0$$

Frequency equation is:

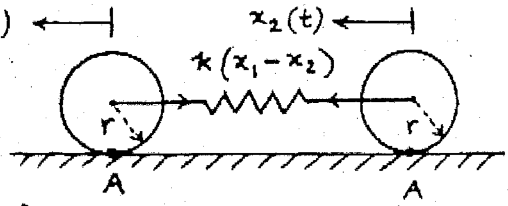
$$(-m_1\omega^2 + k)\left(-\omega^2 m_2 - \omega^2\frac{J_0}{r^2} + k\right) - k^2 = 0$$

or

$$\omega^4\left(m_1 m_2 + \frac{m_1 J_0}{r^2}\right) - \omega^2\left(m_1 k + m_2 k + \frac{k J_0}{r^2}\right) = 0$$

$$\omega_1 = 0, \qquad \omega_2 = \left\{\left(m_1 k + m_2 k + \frac{k J_0}{r^2}\right) \Big/ \left(m_1 m_2 + \frac{m_1 J_0}{r^2}\right)\right\}^{1/2}$$

(5.54) Equivalent mass of each cylinder for translatory motion, referred to point A, is



$$m_{eq} = \frac{J_A}{r^2} = \frac{\frac{m r^2}{2} + m r^2}{r^2} = \frac{3m}{2} = (m_1)_{eq} = (m_2)_{eq}$$

Equations of motion:

$$(m_1)_{eq}\, \ddot{x}_1 + k(x_1 - x_2) = 0$$

$$(m_2)_{eq}\, \ddot{x}_2 - k(x_1 - x_2) = 0$$

Frequency equation:

$$\begin{vmatrix} -(m_1)_{eq}\,\omega^2 + k & -k \\ -k & -(m_2)_{eq} + k \end{vmatrix} = 0$$

or $$(m_1)_{eq}(m_2)_{eq}\,\omega^4 - \omega^2\left[(m_1)_{eq}\,k + (m_2)_{eq}\,k\right] = 0$$

$$\omega_1 = 0, \qquad \omega_2 = \sqrt{\frac{(m_1)_{eq}\,k + (m_2)_{eq}\,k}{(m_1)_{eq}\,(m_2)_{eq}}} = \sqrt{\frac{4k}{3m}}$$

(5.55) For harmonic motion $x_i(t) = X_i \cos(\omega t + \phi)\ ;\ i = 1, 2$
given equations lead to

$$(-\omega^2 a_1 + b_1) X_1 + c_1 X_2 = 0$$

$$b_2 X_1 + (-\omega^2 a_2 + c_2) X_2 = 0$$

Frequency equation is

$$\begin{vmatrix} -\omega^2 a_1 + b_1 & c_1 \\ b_2 & -\omega^2 a_2 + c_2 \end{vmatrix} = 0$$

or $$\omega^4 (a_1 a_2) - \omega^2 (a_1 c_2 + b_1 a_2) + (b_1 c_2 - c_1 b_2) = 0$$

Condition for degeneracy is: $b_1 c_2 - c_1 b_2 = 0$

(5.56) Equations of motion:

$$J_1 \ddot{\theta}_1 + k_t \theta_1 - k_t \theta_2 = 0$$

$$J_2 \ddot{\theta}_2 + k_t \theta_2 - k_t \theta_1 = 0$$

For $\theta_i(t) = \Theta_i \cos(\omega t + \phi)\ ;\ i = 1, 2,$

$$\begin{bmatrix} -\omega^2 J_1 + k_t & -k_t \\ -k_t & -\omega^2 J_2 + k_t \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is $$\omega^4 (J_1 J_2) - \omega^2 (J_1 k_t + J_2 k_t) = 0$$

$$\omega_1 = 0, \qquad \omega_2 = \sqrt{\frac{k_t (J_1 + J_2)}{J_1 J_2}}$$

Amplitude ratios:

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2 J_1 + k_t}{k_t} = 1$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2 J_1 + k_t}{k_t} = -\frac{J_1}{J_2}$$

General solution is given by equations similar to Eqs. (5-15). With $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$, we obtain from equations similar to Eqs. (5·18):

$$\Theta_1^{(1)} = \frac{1}{r_2 - r_1}\{r_2\,\theta_1(0) - \theta_2(0)\} = -\left(\frac{J_2}{J_1+J_2}\right)\left[-\frac{J_1}{J_2}\,\theta_1(0) - \theta_2(0)\right]$$

$$= \left\{\frac{J_1\,\theta_1(0) + J_2\,\theta_2(0)}{J_1 + J_2}\right\}$$

$$\Theta_1^{(2)} = \frac{1}{r_2 - r_1}\{-r_1\,\theta_1(0) + \theta_2(0)\} = -\left(\frac{J_2}{J_1+J_2}\right)\left[\frac{J_1\,\theta_1(0)}{J_2} + \theta_2(0)\right]$$

$$= -\left\{\frac{J_1\,\theta_1(0) + J_2\,\theta_2(0)}{J_1 + J_2}\right\}$$

$$\phi_1 = \phi_2 = 0$$

$$\theta_1(t) = \Theta_1^{(1)} \cos 0t + \Theta_1^{(2)} \cos \omega_2 t = \Theta_1^{(1)} + \Theta_1^{(2)} \cos \omega_2 t$$

$$\theta_2(t) = \Theta_1^{(1)} - \frac{J_1}{J_2}\,\Theta_1^{(2)} \cos \omega_2 t$$

5.57 When $k_{t1} = 0$, the system becomes identical to the system of problem 5.56 with $k_t = k_{t1}$. Normal modes are given by

$$\vec{\Theta}^{(1)} = \begin{Bmatrix} \Theta_1^{(1)} \\ \Theta_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \Theta_1^{(1)} \quad ; \quad \vec{\Theta}^{(2)} = \begin{Bmatrix} \Theta_1^{(2)} \\ \Theta_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -\left(\frac{J_1}{J_2}\right) \end{Bmatrix} \Theta_1^{(2)}$$

Equations of motion can be rewritten as

$$\ddot{\theta}_1 + \frac{k_t}{J_1}(\theta_1 - \theta_2) = 0 \qquad ---- \quad (E_1)$$

$$\ddot{\theta}_2 - \frac{k_t}{J_2}(\theta_1 - \theta_2) = 0 \qquad ---- \quad (E_2)$$

Subtracting (E_2) from (E_1) gives

$$(\ddot{\theta}_1 - \ddot{\theta}_2) + (\theta_1 - \theta_2)\left(\frac{k_t}{J_1} + \frac{k_t}{J_2}\right) = 0 \qquad ---- \quad (E_3)$$

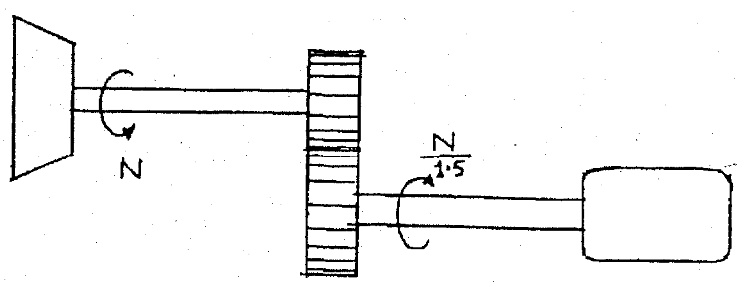
Defining $\alpha = \theta_1 - \theta_2$, (E_3) can be written as

$$\ddot{\alpha} + \left(\frac{k_t}{J_1} + \frac{k_t}{J_2}\right)\alpha = 0 \qquad ---- \quad (E_4)$$

This is a single equation for which the natural frequency is

$$\omega = \sqrt{\left(\frac{k_t}{J_1} + \frac{k_t}{J_2}\right)} = \sqrt{\frac{k_t\,(J_1 + J_2)}{J_1\,J_2}} \equiv \omega_2 \text{ of problem 5.45.}$$

5.58



Since the length of shaft 1 is small and its diameter large, it will be very rigid and hence the turbine and gear 1 are assumed to be rigidly connected. This helps in modeling the system as a two d.o.f. system.

$$J_{01} = J_{turbine} + J_{gear1} + \frac{J_{gear2}}{1.5^2} = 3000 + 500 + (1000/2.25) = 3944.4444 \text{ kg–m}^2$$

$$k_{t2} = \left(\frac{GJ}{\ell}\right)_{shaft2} = \frac{(80\ (10^9))\left(\frac{\pi}{32}\ (0.1^4)\right)}{1} = 7.854\ (10^5) \text{ N/m}$$

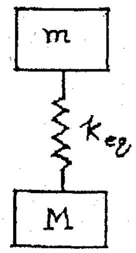
$$J_{02} = J_{generator} = 2000 \text{ kg–m}^2$$

System is a semi-definite system. Its natural frequencies are given by (see Eq. (5.40)):

$$\omega_1 = 0$$

$$\omega_2 = \sqrt{\frac{k_{t2}\ (J_{01} + J_{02})}{J_{01}\ J_{02}}} = \sqrt{\frac{(78.54\ (10^4))\ (5944.4444)}{(3944.4444)\ (2000)}} = 24.3273 \text{ rad/sec}$$

(5.59) Natural frequencies are given by Eq. (5.40):



$$\omega_1 = 0 \quad ; \quad \omega_2 = \sqrt{\frac{k\ (m_1 + m_2)}{m_1\ m_2}} = \sqrt{\frac{6\ k\ (m + M)}{m\ M}}$$

Assumption: Balloon is a point mass.

$$k_{eq} = 12\ k \cos^2 45^\circ = 6\ k$$

(5.60) *Equations of motion :*

$$J_1 \ddot{\theta}_1 + c_{t1} \dot{\theta}_1 - c_{t2}(\dot{\theta}_2 - \dot{\theta}_1) + k_{t1} \theta_1 - k_{t2}(\theta_2 - \theta_1) = 0$$

$$J_2 \ddot{\theta}_2 + c_{t2}(\dot{\theta}_2 - \dot{\theta}_1) + k_{t2}(\theta_2 - \theta_1) = 0$$

i.e.,

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} c_{t1} + c_{t2} & -c_{t2} \\ -c_{t2} & c_{t2} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k_{t1} + k_{t2} & -k_{t2} \\ -k_{t2} & k_{t2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{--- } (E_1)$$

Let $\theta_j(t) = X_j\ e^{st}$ ----- (E_2)

Eqs. (E_1) yield :

$$\left(\begin{bmatrix} J_1 s^2 & 0 \\ 0 & J_2 s^2 \end{bmatrix} + \begin{bmatrix} (c_{t1} + c_{t2}) s & -c_{t2} s \\ -c_{t2} s & c_{t2} s \end{bmatrix} + \begin{bmatrix} (k_{t1} + k_{t2}) & -k_{t2} \\ -k_{t2} & k_{t2} \end{bmatrix} \right) \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{st} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{---- } (E_3)$$

The characteristic equation becomes:

$$\begin{vmatrix} J_1 s^2 + (c_{t1} + c_{t2}) s + k_{t1} + k_{t2} & -(c_{t2} s + k_{t2}) \\ -(c_{t2} s + k_{t2}) & J_2 s^2 + c_{t2} s + k_{t2} \end{vmatrix} = 0$$

i.e.,

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0 \qquad \cdots (E_4)$$

where

$$a_0 = J_1 J_2$$

$$a_1 = J_1 c_{t2} + J_2 (c_{t1} + c_{t2})$$

$$a_2 = J_1 k_{t2} + c_{t2} (c_{t1} + c_{t2}) + J_2 (k_{t1} + k_{t2}) - c_{t2}^2$$

$$a_3 = k_{t2} (c_{t1} + c_{t2}) + c_{t2} (k_{t1} + k_{t2}) - 2 c_{t2} k_{t2}$$

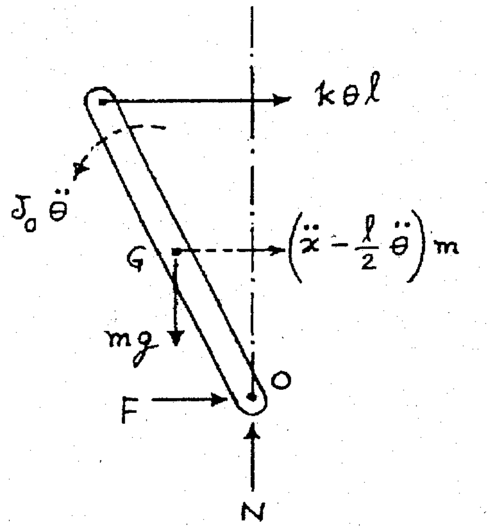
$$a_4 = k_{t2} (k_{t1} + k_{t2}) - k_{t2}^2$$

For the stability of the system, the conditions derived in section 5.8 are applicable:

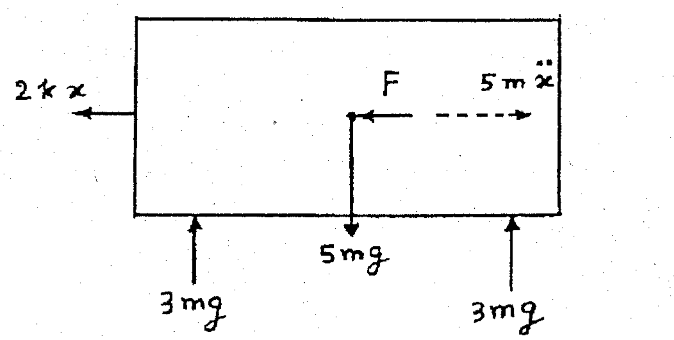
$$a_i > 0 \quad ; \quad i = 0, 1, 2, 3, 4$$

$$a_1 a_2 a_3 - a_1^2 a_4 - a_0 a_3^2 > 0$$

5.61



Free body diagram of bar



Free body diagram of trailer

Equations of motion of bar:

$$m\,(\ddot{x} - \frac{\ell}{2}\,\ddot{\theta}) = k\,\theta\,\ell + F \tag{1}$$

$$J_0\,\ddot{\theta} - m\,(\ddot{x} - \frac{\ell}{2}\,\ddot{\theta})\,\frac{\ell}{2} = -k\,\theta\,\ell\,(\ell) + m\,g\,\frac{\ell}{2}\sin\theta \tag{2}$$

Equation of motion of trailer:

$$5\,m\,\ddot{x} = -F - 2\,k\,x \quad \text{or} \quad F = -2\,k\,x - 5\,m\,\ddot{x} \tag{3}$$

Equations (1) and (2) can be rewritten as:

$$m\left(\ddot{x}-\frac{\ell}{2}\,\ddot{\theta}\right)-k\,\theta\,\ell+2\,k\,x+5\,m\,\ddot{x}=0 \tag{5}$$

$$J_0\,\ddot{\theta}-\frac{m\,\ell}{2}\,(\ddot{x}-\frac{\ell}{2}\,\ddot{\theta})+k\,\theta\,\ell^2-\frac{m\,g\,\ell\,\theta}{2}=0 \tag{6}$$

$$\text{where}\quad J_0=\frac{1}{3}\,m\,\ell^2 \tag{7}$$

Equations (5) and (6) can be expressed in matrix form as:

$$\begin{bmatrix} 6\,m & -\dfrac{m\,\ell}{2} \\ -\dfrac{m\,\ell}{2} & \dfrac{7}{12}\,m\,\ell^2 \end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{\theta}\end{Bmatrix}+\begin{bmatrix} 2\,k & -k\,\ell \\ 0 & (k\,\ell^2-\dfrac{1}{2}\,m\,g\,\ell)\end{bmatrix}\begin{Bmatrix}x\\ \theta\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{8}$$

Assuming a solution of the form:

$$x(t)=X\,e^{s\,t}\quad\text{and}\quad\theta(t)=\Theta\,e^{s\,t} \tag{9}$$

Eq. (8) can be expressed as:

$$\left[s^2\begin{bmatrix} 6\,m & -\dfrac{m\,\ell}{2} \\ -\dfrac{m\,\ell}{2} & \dfrac{m\,\ell^2}{3}\end{bmatrix}+\begin{bmatrix} 2\,k & -k\,\ell \\ 0 & -\left(\dfrac{m\,g\,\ell}{2}-k\,\ell^2\right)\end{bmatrix}\right]\begin{Bmatrix}X\\ \Theta\end{Bmatrix}e^{s\,t}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{10}$$

By setting the determinant of the coefficient matrix in Eq. (10) equal to zero, we obtain:

$$\begin{vmatrix} (6\,m\,s^2+2\,k) & -\left(\dfrac{m\,\ell\,s^2}{2}+k\,\ell\right) \\ -\left(\dfrac{m\,\ell\,s^2}{2}\right) & \left(\dfrac{m\,\ell^2\,s^2}{3}-\dfrac{m\,g\,\ell}{2}+k\,\ell^2\right)\end{vmatrix}=0 \tag{11}$$

which, upon expansion, gives:

$$\left(\frac{7}{4}\,m^2\,\ell^2\right)s^4+\left(\frac{37}{6}\,m\,k\,\ell^2-3\,m^2\,g\,\ell\right)s^2+\left(-m\,k\,g\,\ell+2\,k^2\,\ell^2\right)=0 \tag{12}$$

A comparison of Eq. (12) with Eq. (5.43) gives:

$$a_0=\frac{7}{4}\,m^2\,\ell^2$$

$$a_1=0$$

$$a_2=\frac{37}{6}\,m\,k\,\ell^2-3\,m^2\,g\,\ell$$

$$a_3=0$$

$$a_4=2\,k^2\,\ell^2-m\,k\,g\,\ell$$

Conditions for the stability of the system:

1. All coefficients a_i must be positive:

$$a_2 \geq 0 \text{ or } k \geq \frac{18}{37}\frac{m\,g}{\ell}$$
$$a_4 \geq 0 \text{ or } k \geq \frac{1}{2}\frac{m\,g}{\ell}$$

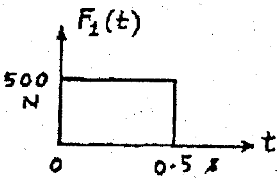
2.

$$a_1\, a_2\, a_3 > a_0\, a_3^2 + a_4\, a_1^2$$

This is not applicable since both sides of the inequality are zero.

Thus the condition for stability is: $k \geq \frac{1}{2}\frac{m\,g}{\ell}$.

(5.62) Equations of motion are (Eqs. (5.1) and (5.2))



$$2m\,\ddot{x}_1 + 3k\,x_1 - 2k\,x_2 = F_1(t)$$
$$m\,\ddot{x}_2 - 2k\,x_1 + 3k\,x_2 = 0$$

Hence $[m] = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} 20 & 0 \\ 0 & 10 \end{bmatrix}$, $[k] = \begin{bmatrix} 3k & -2k \\ -2k & 3k \end{bmatrix} = \begin{bmatrix} 6000 & -4000 \\ -4000 & 6000 \end{bmatrix}$,

$$\vec{F} = \begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix}$$

Frequency equation is $\left| -\omega^2 [m] + [k] \right| = \omega^4 - 900\,\omega^2 + 100\,000 = 0$

Hence $\omega_1 = 11.3949$ rad/s, $\omega_2 = 27.7517$ rad/s

and $\tau_1 = 0.5514$ s , $\tau_2 = 0.2264$ s.

We select $\Delta t = 0.02$ s and use central difference method for numerical solution (see chapter 11 for details).

The main program which calls CDIFF, the subroutine EXTFUN and the output are given below.

```
C =====================================================================
C
C PROGRAM
C MAIN PROGRAM WHICH CALLS CDIFF
C
C =====================================================================
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      PEAL M(2,2),K(2,2),MC(2,2),MK(2,2),MCI(2,2),MMC(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),XM1(2),F(2),R(2),RR(2),
     2 XMK(2),XMI(2),XM2(2),XP1(2),ZA(2),ZB(2),ZC(2),LA(2),LB(2,2),
     3 S(2),X(50,2),XD(50,2),XDD(50,2)
      DATA N,NSIEP,NSTEP1,DELT/2,49,50,0.02/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/20.0,0.0,0.0,10.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6000.0,-4000.0,-4000.0,6000.0/
C END OF PROBLEM-DEPENDENT DATA
      CALL CDIFF (M,C,K,XI,XDI,XDDI,N,NSTEP,DELT,F,R,RR,XM1,XM2,XP1,
     2 MC,MK,MCI,XMK,MMC,XMI,ZA,ZB,ZC,LA,LB,S,X,XD,XDD,NSTEP1)
```

```
      WRITE (13,10)
10    FORMAT (//,38H SOLUTION BY CENTRAL DIFFERENCE METHOD,/)
      WRITE (13,20) N,NSTEP,DELT
20    FORMAT (12H GIVEN DATA:,/,3H N=,I5,4X,7H NSTEP=,I5,4X,6H DELT=,
     2 E15.8,/)
      WRITE (13,30)
30    FORMAT (10H SOLUTION:,//,5H STEP,3X,5H TIME,3X,7H X(I,1),3X,
     2 8H XD(I,1),2X,9H XDD(I,1),4X,7H X(I,2),3X,8H XD(I,2),2X,
     3 9H XDD(I,2),/)
      DO 40, I=1,NSTEP1
      TIME=REAL(I-1)*DELT
40    WRITE (13,50) I,TIME,X(I,1),XD(I,1),XDD(I,1),X(I,2),XD(I,2),XDD(
     2 I,2)
50    FORMAT (1X,I4,F8.4,6(1X,E10.4))
      STOP
      END
C =====================================================================
C
C SUBROUTINE EXTFUN
C THIS SUBROUTINE IS PROBLEM-DEPENDENT
C
C =====================================================================
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=0.0
      F(2)=0.0
      IF (TIME .LE. 0.5) F(1)=500.0
      RETURN
      END
```

```
SOLUTION BY CENTRAL DIFFERENCE METHOD

GIVEN DATA:
N=    2     NSTEP=   49      DELT= 0.20000000E-01

SOLUTION:
```

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.2500E+02	0.0000E+00	0.0000E+00	0.0000E+00
2	0.0200	0.5000E-02	0.0000E+00	0.2500E+02	0.0000E+00	0.0000E+00	0.0000E+00
3	0.0400	0.1940E-01	0.4850E+00	0.2350E+02	0.8000E-03	0.2000E-01	0.2000E+01
4	0.0600	0.4154E-01	0.9134E+00	0.1934E+02	0.4512E-02	0.1128E+00	0.7280E+01
5	0.0800	0.6905E-01	0.1241E+01	0.1344E+02	0.1379E-01	0.3247E+00	0.1391E+02
6	0.1000	0.9938E-01	0.1446E+01	0.7043E+01	0.3080E-01	0.6572E+00	0.1935E+02
7	0.1200	0.1302E+00	0.1530E+01	0.1347E+01	0.5632E-01	0.1063E+01	0.2127E+02
8	0.1400	0.1600E+00	0.1515E+01	-.2810E+01	0.8917E-01	0.1459E+01	0.1831E+02
9	0.1600	0.1877E+00	0.1436E+01	-.5164E+01	0.1262E+00	0.1747E+01	0.1050E+02
10	0.1800	0.2129E+00	0.1323E+01	-.6059E+01	0.1630E+00	0.1846E+01	-.6577E+00
	⋮						
46	0.9000	0.2345E+00	-.1023E+01	-.8737E+01	0.1991E+00	-.5925E+00	-.2714E+02
47	0.9200	0.2101E+00	-.1165E+01	-.5533E+01	0.1715E+00	-.1120E+01	-.2565E+02
48	0.9400	0.1842E+00	-.1258E+01	-.3717E+01	0.1365E+00	-.1566E+01	-.1889E+02
49	0.9600	0.1571E+00	-.1325E+01	-.2966E+01	0.9808E-01	-.1837E+01	-.8195E+01
50	0.9800	0.1290E+00	-.1379E+01	-.2515E+01	0.6131E-01	-.1879E+01	0.3993E+01

(5.63) (a) Frequency equation is

$$\left|-\omega^2[m]+[k]\right| = \left|-\omega^2\begin{bmatrix}0.2 & 0\\ 0 & 0.2\end{bmatrix}+\begin{bmatrix}36 & -18\\ -18 & 18\end{bmatrix}\right| = \omega^4 - 270\,\omega^2 + 8100 = 0$$

```
C ==========================================================================
C
C PROGRAM 8
C MAIN PROGRAM FOR CALLING THE SUBROUTINE QUART
C
C ==========================================================================
C SOLUTION OF:  A(1)*(X**4)+A(2)*(X**3)+A(3)*(X**2)+A(4)*X+A(5)=0
      DIMENSION A(5),RR(4),RI(4)
C FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
      DATA A/1.0,0.0,-270.0,0.0,8100.0/
C END OF PROBLEM-DEPENDENT DATA
      WRITE (26,10) (A(I),I=1,5)
 10   FORMAT (//,31H SOLUTION OF A QUARTIC EQUATION,//,6H DATA:,/,
     2 7H A(1) =,E15.6,/,7H A(2) =,E15.6,/,7H A(3) =,E15.6,/,
     3 7H A(4) =,E15.6,/,7H A(5) =,E15.6,/)
      CALL QUART (A,RR,RI)
      WRITE (26,20)
 20   FORMAT (/,7H ROOTS:,//,9H ROOT NO.,3X,10H REAL PART,5X,
     2 15H IMAGINARY PART,/)
      DO 30 I=1,4
 30   WRITE (26,40) I,RR(I),RI(I)
 40   FORMAT (I5,3X,E15.6,3X,E15.6)
      STOP
      END
```

```
SOLUTION OF A QUARTIC EQUATION

DATA:
A(1) =   0.100000E+01
A(2) =   0.000000E+00
A(3) =  -0.270000E+03
A(4) =   0.000000E+00
A(5) =   0.810000E+04

ROOTS:

ROOT NO.    REAL PART       IMAGINARY PART

    1     -0.153500E+02      0.000000E+00
    2     -0.586319E+01      0.000000E+00
    3      0.586319E+01      0.000000E+00
    4      0.153500E+02      0.000000E+00
```

(b) $\left(-0.2\,\omega_j^2+36\right)X_j^{(1)} - 18\,X_j^{(2)} = 0 \;;\quad j=1,2$

Writing $X_j^{(2)} = \left(\frac{-0.2\,\omega_j^2+36}{18}\right)X_j^{(1)} \equiv r_j\,X_j^{(1)}$,

$r_1 = \left\{-0.2\,(5.86319)^2+36\right\}/18 = 1.6180334$

$r_2 = \left\{-0.2\,(15.35)^2+36\right\}/18 = -0.6180278$

Eqs. (5.18) give $X_1^{(1)} = 1.44722$, $X_1^{(2)} = 0.55279$, $\phi_1 = \phi_2 = 0$

Displacements of masses m_1 and m_2 are given by Eqs. (5.15):

$x_1(t) = 1.44722\cos(5.86319\,t) + 0.55279\cos(15.35\,t)$

$x_2(t) = 2.34165\cos(5.86319\,t) - 0.34164\cos(15.35\,t)$

5.64

```
C ======================================================================
C
C PROBLEM 5.64
C
C ======================================================================
      DIMENSION C(2,2),FZ(2)
      REAL K(2,2),M(2,2)
      COMPLEX Z(2,2),X(2),AA,BB,DEN
C INPUT DATA
      DATA M/0.1,0.0,0.0,0.1/
      DATA C/1.0,0.0,0.0,0.0/
      DATA K/40.0,-20.0,-20.0,20.0/
      DATA FZ/1.0,2.0/
      OMF=5.0
C END OF INPUT DATA
      DO 10 I=1,2
      DO 10 J=1,2
      A=-(OMF**2)*M(I,J)+K(I,J)
      B=OMF*C(I,J)
 10   Z(I,J)=CMPLX(A,B)
      DEN=Z(1,1)*Z(2,2)-Z(1,2)*Z(1,2)
      AA=Z(2,2)*CMPLX(FZ(1),0.0)
      BB=Z(1,2)*CMPLX(FZ(2),0.0)
      X(1)=(AA-BB)/DEN
      AA=Z(1,1)*CMPLX(FZ(2),0.0)
      BB=Z(1,2)*CMPLX(FZ(1),0.0)
      X(2)=(AA-BB)/DEN
      PRINT 20, X(1),X(2)
 20   FORMAT (//,2X,25H SOLUTION OF PROBLEM 5.49,//,2X,
     2 7H X(1) =,2E15.8,/,2X,7H X(2) =,2E15.8,/)
      PRINT 30, OMF
 30   FORMAT (2X,6H OMF =,E15.8)
      STOP
      END
SOLUTION OF PROBLEM 5.64

X(1) = 0.20095898E+00-0.68620138E-01
X(2) = 0.34395310E+00-0.78423016E-01

OMF = 0.50000000E+01
```

5.65 The system shown in Fig. A can be drawn in equivalent form as shown in Fig. B where both pulleys have the same radius, r_1.

The equivalent mass moment of inertia of pulley 2 can be computed in different speed ratios as:

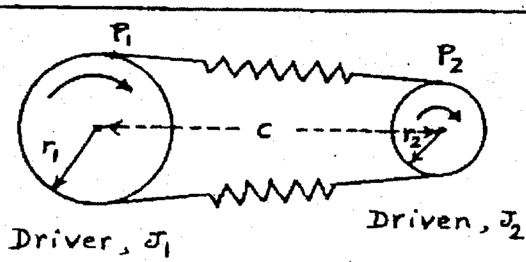
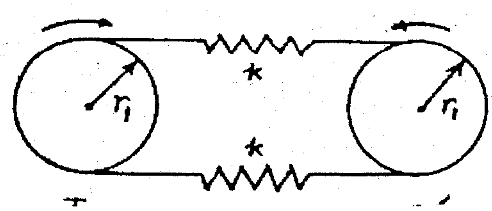


Fig. A



$$J_2' = J_2\,(\text{speed ratio})^2$$
$$= J_2\left(\frac{150}{350}\right)^2 ;\quad J_2\left(\frac{250}{350}\right)^2 ;$$
$$J_2\left(\frac{450}{350}\right)^2 ;\quad J_2\left(\frac{750}{350}\right)^2$$

or
$$J_2' = 0.1837\,J_2 ;\quad 0.5102\,J_2 ;\quad 1.6531\,J_2 ;\quad 4.5918\,J_2$$

Stiffness of the belt (on each side) is given by
$$k = \frac{A E}{l}$$
where A = cross-sectional area of belt, E = Young's modulus and l = length of the belt. Length of the belt (distance $P_1 P_2$ in Fig. A) is given by
$$l = \frac{1}{2}\left[4c^2 - (D-d)^2\right]^{\frac{1}{2}}$$
In this example, $c = 5$ m, $D = 1$ m, $d = 0.25$ m and hence
$$l = \frac{1}{2}\left[4(5)^2 - (1-0.25)^2\right]^{1/2} = 4.9859 \text{ m}$$
$$\therefore\quad k = \frac{A\,(10^{10})}{4.9859} = 2.0057 \times 10^9\, A \quad N/m$$

Equation of motion:
$$J_1\ddot{\theta}_1 + k_t(\theta_1+\theta_2) = 0 \;\Rightarrow\; J_1\ddot{\theta}_1 + k_t\,\theta_1\left(1 + \frac{J_1}{J_2'}\right) = 0$$
$$J_2\ddot{\theta}_2 + k_t(\theta_1+\theta_2) = 0 \;\Rightarrow\; J_2\ddot{\theta}_2 + k_t\,\theta_2\left(\frac{J_2'}{J_1} + 1\right) = 0$$
$$\therefore\quad \omega_n = \sqrt{k_t\left(\frac{J_1 + J_2'}{J_1 J_2'}\right)}$$
where $k_t = 2 k r_1^2$ (see solution of problem 2.51).

Here $J_1 = 0.1$ kg-m^2 and $J_2 = 0.2$ kg-m^2. In order for the natural frequency ω_n to be away from the speeds 150, 250, 350, 450 and 750 rpm {or, 15.708, 26.180, 36.652, 47.124 and 78.540 rad/sec},
$$\omega_n \le 15.708 \text{ rad/sec}$$
$$\omega_n \ge 78.540 \text{ rad/sec}$$
Since ω_n involves A (through k_t), it can be determined from the above inequalities.

5.66 Velocity of tup before impact is given by:

$$\frac{1}{2}\ m_{tup}\ v^2 = m_{tup}\ g\ h \quad \text{or} \quad v = \sqrt{2\ g\ h} = \sqrt{2\ (9.81)\ (2)} = 6.2642\ \text{m/sec}$$

(a) Impact is inelastic:

Conservation of momentum leads to:

$$m_{tup}\ v_{tup} + m_{anvil}\ (0) = (m_{tup} + m_{anvil})\ v_0$$

$$\text{or} \quad v_0 = \frac{(1000)\ (6.2642)}{(1000 + 5000)} = 1.0440\ \text{m/sec}$$

(b) Natural frequencies:

$$\omega_{2,1}^2 = \frac{k_1 + k_2}{2\ m_1} + \frac{k_2}{2\ m_2} \pm \sqrt{\frac{1}{4}\left[\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right]^2 - \frac{k_1\ k_2}{m_1\ m_2}}$$

Thus the natural frequency requirement can be stated as:

$$f_1^2 = \frac{\omega_1^2}{(2\ \pi)^2}$$

$$= \frac{1}{(2\ \pi)^2}\left\{\frac{k_1 + k_2}{50000} + \frac{k_2}{10000} - \sqrt{\frac{1}{4}\left[\frac{k_1 + k_2}{25000} + \frac{k_2}{5000}\right]^2 - \frac{k_1\ k_2}{125\ (10^8)}}\right\} > (5^2) \tag{1}$$

(c) Free vibration response:

Initial conditions:

$$x_1(0) = x_2(0) = \dot{x}_1(0) = 0, \quad \dot{x}_2(0) = v_0 = 1.0440\ \text{m/sec}$$

Maximum forces in the springs:

$$F_1 = k_1\ x_1 \big|_{max} \tag{2}$$

$$F_2 = k_2\ (x_2 - x_1) \big|_{max} \tag{3}$$

For a helical spring, the shear stress (τ) under an axial force F is given by:

$$\tau = k_s\ \frac{8\ F\ D}{\pi\ d^3} \tag{4}$$

where k_s = shear stress correction factor $= \dfrac{2D + d}{2\ D}$, D = mean coil diameter, and d = wire diameter.

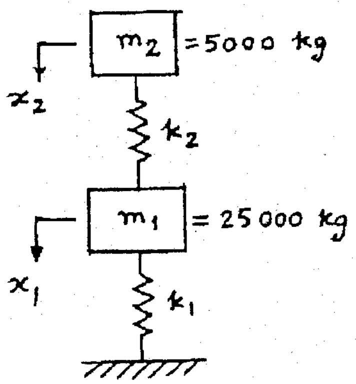
Ref: J. E. Shigley and C. R. Mischke, "Mechanical Engineering Design," 5th Ed., McGraw-Hill, New York, 1989.

Since stress is to be less than the yield stress witth a factor of safety of 1.5, we have

$$\tau_1 \leq \frac{\tau_{\text{yield}}}{1.5} \tag{5}$$

$$\tau_2 \leq \frac{\tau_{\text{yield}}}{1.5} \tag{6}$$

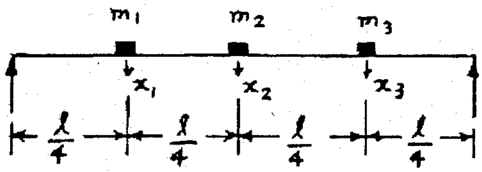
where τ_1 and τ_2 denote the shear stresses induced in the springs k_1 and k_2, respectively, and τ_{yield} is the shear stress corresponding to the yield stress of the material.



Chapter 7

Determination of Natural Frequencies and Mode Shapes

7.1 From Example 6.6,



$$a_{11} = a_{33} = \frac{3}{256}\,\frac{l^3}{EI}$$

$$a_{22} = \frac{1}{48}\,\frac{l^3}{EI}$$

(a) Eq. (7.6) gives

$$\frac{1}{\omega_1^2} \simeq \frac{m l^3}{EI}\left(\frac{3\times 5}{256} + \frac{1}{48}\times 1 + \frac{3\times 5}{256}\right) = \frac{106}{768}\,\frac{m l^3}{EI} = 0.13802\,\frac{m l^3}{EI}$$

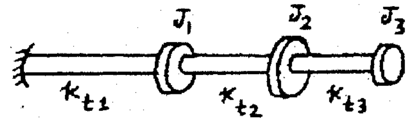
$$\omega_1 \simeq 2.6917\sqrt{\frac{EI}{m l^3}}$$

(b)

$$\frac{1}{\omega_1^2} \simeq \frac{m l^3}{EI}\left(\frac{3}{256} + \frac{1\times 5}{48} + \frac{3}{256}\right) = \frac{98}{768}\,\frac{m l^3}{EI} = 0.12760\,\frac{m l^3}{EI}$$

$$\omega_1 \simeq 2.7994\sqrt{\frac{EI}{m l^3}}$$

7.2 Flexibility influence coefficients:



a_{11} = rotation of J_1 when a unit torque is applied to J_1 = $1/k_{t1}$

a_{22} = rotation of J_2 when a unit torque is applied to J_2

$$= \frac{1}{k_{teq}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}}$$

a_{33} = rotation of J_3 when a unit torque is applied to J_3

$$= \frac{1}{k_{teq}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}} + \frac{1}{k_{t3}}$$

(a) Eq. (7.6) gives $\frac{1}{\omega_1^2} \simeq a_{11} J_1 + a_{22} J_2 + a_{33} J_3 = \frac{J_0}{k_t}(1+2+3)$

$$\omega_1 \simeq 0.4082\sqrt{k_t/J_0}$$

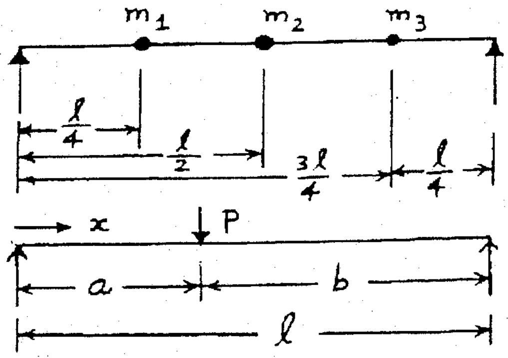
(b)

$$\frac{1}{\omega_1^2} \simeq \frac{J_1}{k_{t1}} + J_2\left(\frac{1}{k_{t1}} + \frac{1}{k_{t2}}\right) + J_3\left(\frac{1}{k_{t1}} + \frac{1}{k_{t2}} + \frac{1}{k_{t3}}\right)$$

$$\simeq \frac{J_0}{k_t} + 2J_0\left(\frac{1}{k_t} + \frac{1}{2k_t}\right) + 3J_0\left(\frac{1}{k_t} + \frac{1}{2k_t} + \frac{1}{3k_t}\right) = \frac{9.5\,J_0}{k_t}$$

$$\omega_1 \simeq 0.3244\sqrt{k_t/J_0}$$

7.3 $m_1 = m, \quad m_2 = 2m, \quad m_3 = 3m;$



$l_1 = l_2 = l_3 = l_4 = l/4$

$$w(x) = \frac{P b x}{6EIl}\left(l^2 - b^2 - x^2\right); \quad 0 \le x \le a$$

$$= -\frac{Pa(l-x)}{6EIl}\left(a^2 + x^2 - 2lx\right); \quad a \le x \le l$$

Deflection due to weight of m_1: $(P = mg)$

At location of m_1 $\left(x = \frac{l}{4},\ b = \frac{3l}{4},\ l = l\right)$

$$w_1' = \frac{(mg)\left(\frac{3l}{4}\right)\left(\frac{l}{4}\right)}{6EIl}\left\{l^2 - \frac{9}{16}l^2 - \frac{1}{16}l^2\right\} = \frac{3mgl^3}{256\ EI}$$

At location of m_2 $\left(a = \frac{l}{4},\ x = \frac{l}{2},\ l = l\right)$

$$w_2' = -\frac{(mg)\left(\frac{l}{4}\right)\left(l - \frac{l}{2}\right)}{6EIl}\left(\frac{l^2}{16} + \frac{l^2}{4} - l^2\right) = \frac{11\ mg\ l^3}{768\ EI}$$

At location of m_3 $\left(a = \frac{l}{4},\ x = \frac{3l}{4},\ l = l\right)$

$$w_3' = -\frac{(mg)\left(\frac{l}{4}\right)\left(l - \frac{3l}{4}\right)}{6EIl}\left(\frac{l^2}{16} + \frac{9}{16}l^2 - \frac{6}{4}l^2\right) = \frac{7mgl^3}{768EI}$$

Deflection due to weight of m_2: $(P = 2mg)$

At location of m_1 $\left(x = \frac{l}{4},\ b = \frac{l}{2},\ l = l\right)$

$$w_1'' = \frac{(2mg)\left(\frac{l}{2}\right)\left(\frac{l}{4}\right)}{6EIl}\left(l^2 - \frac{l^2}{4} - \frac{1}{16}l^2\right) = \frac{11\ mg\ l^3}{384\ EI}$$

At location of m_2 $\left(x = \frac{l}{2},\ b = \frac{l}{2},\ l = l\right)$

$$w_2'' = \frac{(2mg)\left(\frac{l}{2}\right)\left(\frac{l}{2}\right)}{6EIl}\left(l^2 - \frac{l^2}{4} - \frac{l^2}{4}\right) = \frac{mg\ l^3}{24\ EI}$$

At location of m_3 $\left(x = \frac{3l}{4},\ a = \frac{l}{2},\ b = \frac{l}{2},\ l = l\right)$

$$w_3'' = -\frac{(2mg)\left(\frac{l}{2}\right)\left(l - \frac{3l}{4}\right)}{6EIl}\left(\frac{l^2}{4} + \frac{9}{16}l^2 - \frac{6}{4}l^2\right) = \frac{11\ mg\ l^3}{1024\ EI}$$

Deflection due to weight of m_3: $(P = 3mg)$

At location of m_1 $\left(x = \frac{l}{4},\ b = \frac{l}{4},\ l = l\right)$

$$w_1''' = \frac{(3mg)\left(\frac{l}{4}\right)\left(\frac{l}{4}\right)}{6EIl}\left(l^2 - \frac{l^2}{16} - \frac{l^2}{16}\right) = \frac{7mg\ l^3}{256\ EI}$$

At location of m_2 $(x = \frac{l}{2},\ b = \frac{l}{4},\ l = l)$

$$w_2''' = \frac{(3mg)\left(\frac{l}{4}\right)\left(\frac{l}{2}\right)}{6EIl}\left(l^2 - \frac{l^2}{16} - \frac{l^2}{4}\right) = \frac{11\, mg\, l^3}{256\ EI}$$

At location of m_3 $(x = \frac{3l}{4},\ b = \frac{l}{4},\ l = l)$

$$w_3''' = \frac{(3mg)\left(\frac{l}{4}\right)\left(\frac{3l}{4}\right)}{6EIl}\left(l^2 - \frac{1}{16}l^2 - \frac{9}{16}l^2\right) = \frac{9\, mg\, l^3}{256\ EI}$$

Total deflection of masses:

$$w_1 = w_1' + w_1'' + w_1''' = \frac{mg\,l^3}{EI}\left(\frac{3}{256} + \frac{11}{384} + \frac{7}{256}\right) = \frac{13}{192}\ \frac{mg\,l^3}{EI}$$

$$w_2 = w_2' + w_2'' + w_2''' = \frac{mg\,l^3}{EI}\left(\frac{11}{768} + \frac{1}{24} + \frac{11}{256}\right) = \frac{19}{192}\ \frac{mg\,l^3}{EI}$$

$$w_3 = w_3' + w_3'' + w_3''' = \frac{mg\,l^3}{EI}\left(\frac{7}{768} + \frac{11}{1024} + \frac{9}{256}\right) = \frac{169}{3072}\ \frac{mg\,l^3}{EI}$$

$$\omega = \left\{\frac{g\ \frac{m^2 g\, l^3}{EI}\left(\frac{13}{192} + 2\times\frac{19}{192} + 3\times\frac{169}{3072}\right)}{\frac{m^2 g^2 l^6\, m}{E^2 I^2}\left\{\left(\frac{13}{192}\right)^2 + 2\left(\frac{19}{192}\right)^2 + 3\left(\frac{169}{3072}\right)^2\right\}}\right\}^{1/2}$$

$$= 3.5987\sqrt{\frac{EI}{m\,l^3}}$$

7.4 ω_{11} = natural frequency of wing itself $= 20\ Hz = 125.664\ \frac{rad}{sec}$

ω_{22} = natural frequency of weapon attached at the tip of the wing (neglecting the effect of mass of wing)

$$= \sqrt{\frac{k}{m}} = \sqrt{\frac{50000}{12}\left(\frac{386.4}{2000}\right)} = 28.3725\ \frac{rad}{sec}$$

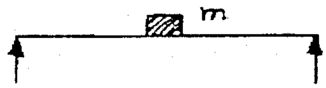
New frequency of vibration of the wing with weapon is given by

$$\frac{1}{\omega_1^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} = \frac{1}{125.664^2} + \frac{1}{28.3725^2}$$

$$= 130.5563 \times 10^{-5}$$

$$\therefore\ \omega_1 = 27.6759\ \frac{rad}{sec} = 4.4047\ Hz$$

7.5 For a simply supported beam, natural frequency is given by (assuming its mass to be concentrated at the middle)



$$\omega_{11} = \sqrt{\frac{k}{m}} \qquad \text{where} \quad k = \frac{48EI}{l^3}$$

Neglecting the mass of beam, if trolley is placed at the middle of girder, its natural frequency is given by

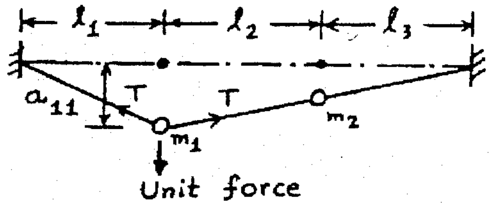
$$\omega_{22} = \sqrt{\frac{k}{10m}}$$

Fundamental natural frequency of the combined system is given by

$$\frac{1}{\omega_1^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} = \frac{m}{k} + \frac{10m}{k} = \frac{11m}{k}$$

$$\therefore \quad \omega_1 = 0.3015\sqrt{\frac{k}{m}} = 30.15\,\% \text{ of the natural frequency of the girder (without the trolley)}$$

7.6 Flexibility coefficients:



Let T = tension in string

a_{11} = deflection of m_1 due to unit force applied to m_1

Unit force = sum of components of tension in vertical direction

i.e. $$1 = T\left(\frac{a_{11}}{l_1}\right) + T\left(\frac{a_{11}}{l_2 + l_3}\right) = T\,a_{11}\left(\frac{1}{l} + \frac{1}{2l}\right) = \frac{3T\,a_{11}}{2l}$$

$$a_{11} = \frac{2l}{3T} = a_{22} \text{ (by symmetry)}$$

$$\frac{1}{\omega_1^2} \simeq a_{11}m_1 + a_{22}m_2 = \frac{2l}{3T}(m+m) = \frac{4ml}{3T}$$

$$\omega_1 \simeq 0.866\sqrt{\frac{T}{ml}}\ ; \quad \text{Exact solution is } \omega_1 = \sqrt{\frac{T}{ml}} \text{ (Problem 5.18).}$$

7.7 From Example 6.5,

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = k\begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Assuming the mode shape as $\vec{x} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$, Rayleigh's quotient becomes

$$R(\vec{x}) = \omega^2 = \frac{\vec{x}^T [k] \vec{x}}{\vec{x}^T [m] \vec{x}}$$

$$= \frac{(1 \quad 2 \quad 3)\, k \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}{(1 \quad 2 \quad 3)\, m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}} = \frac{1}{6} \frac{k}{m}$$

$$\omega_1 = 0.4082 \sqrt{\frac{k}{m}}$$

Exact value is $\omega_1 = 0.3376 \sqrt{\frac{k}{m}}$ (Problem 6.46).

(7.8) Stiffness matrix:

Give each disc a unit angular displacement holding other discs with zero rotation. Torque required will give stiffness coefficients.

$\theta_1 = 1,\ \theta_2 = \theta_3 = 0:\quad M_{t1} = k_{t1} + k_{t2},\quad M_{t2} = -k_{t2},\quad M_{t3} = 0$

$\theta_2 = 1,\ \theta_1 = \theta_3 = 0:\quad M_{t2} = k_{t2} + k_{t3},\quad M_{t1} = -k_{t2},\quad M_{t3} = -k_{t3}$

$\theta_3 = 1,\ \theta_1 = \theta_2 = 0:\quad M_{t3} = k_{t3},\quad M_{t2} = -k_{t3},\quad M_{t1} = 0$

$$[k] = \begin{bmatrix} k_{t1} + k_{t2} & -k_{t2} & 0 \\ -k_{t2} & k_{t2} + k_{t3} & -k_{t3} \\ 0 & -k_{t3} & k_{t3} \end{bmatrix} = k_t \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

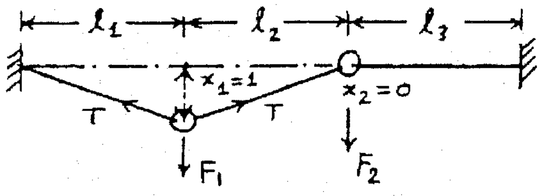
$$[m] = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} = J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Assume the mode shape as $\vec{\Theta} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$

$$R(\vec{\Theta}) = \omega^2 = \frac{(1 \quad 2 \quad 3)\, k_t \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}{(1 \quad 2 \quad 3)\, J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}} = \frac{1}{12} \frac{k_t}{J_0}$$

$$\omega_1 \simeq 0.2887 \sqrt{\frac{k_t}{J_0}}$$

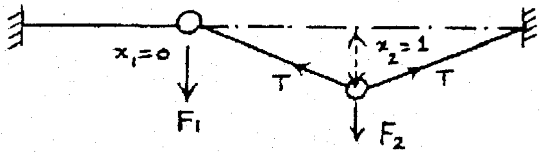
(7.9) Stiffness matrix:



When $x_1 = 1$, $x_2 = 0$:

$$F_1 = k_{11} = \frac{T}{l_1} + \frac{T}{l_2}$$

$$F_2 = k_{21} = -\frac{T}{l_2}$$



When $x_1 = 0$, $x_2 = 1$:

$$F_2 = \frac{T}{l_2} + \frac{T}{l_3}$$

$$F_1 = k_{12} = -\frac{T}{l_2}$$

$$[k] = T \begin{bmatrix} \left(\frac{1}{l_1} + \frac{1}{l_2}\right) & -\frac{1}{l_2} \\ -\frac{1}{l_2} & \left(\frac{1}{l_2} + \frac{1}{l_3}\right) \end{bmatrix} = \frac{T}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Assume the mode shape as $\vec{X} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

$$R(\vec{X}) = \omega^2 = \frac{(1 \quad 2) \frac{T}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}}{(1 \quad 2)\, m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}} = \frac{6}{5} \frac{T}{l m}$$

$$\omega_1 \simeq 1.0954 \sqrt{\frac{T}{l m}}$$

Exact value is $\omega_1 = \sqrt{\frac{T}{l m}}$ (Problem 5.18).

(7.10) From problem 7.9, for $l_1 = l_2 = l_3 = l$,

$$[k] = \frac{T}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

and for $m_1 = m$, $m_2 = 5m$, $[m] = m \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

Let $\vec{X} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

$$R(\vec{X}) = \omega^2 = \frac{(1 \quad 2) \frac{T}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}}{(1 \quad 2)\, m \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}} = \frac{2}{7} \frac{T}{m l}$$

$$\omega_1 \simeq 0.5345 \sqrt{\frac{T}{m l}}$$

(7.11) Apply a unit load to masses m_1 and m_2 along x_1 and x_2, respectively:

$$a_{11} = \frac{1}{2k_1} \quad ; \quad a_{22} = \frac{1}{2k_1} + \frac{1}{2k_2} = \frac{k_2 + k_1}{2 k_1 k_2}$$

Dunkerley's equation gives

$$\frac{1}{\omega_1^2} = a_{11} m_1 + a_{22} m_2 = \frac{m_1}{2k_1} + m_2 \left(\frac{k_2 + k_1}{2 k_1 k_2} \right) \qquad (E_1)$$

Since $k_1 = k_2 = 3EI/h^3 \equiv k$, $m_1 = 2m$, $m_2 = m$ and hence (E_1) gives

$$\omega_1 = \sqrt{\frac{3EI}{2 m h^3}}$$

(7.12) Eq. (7.21) gives, for $r = n$,

$$R(\vec{x}) = \frac{c_n^2 \omega_n^2 + c_n^2 \sum_{i=1}^{n-1} \left(\frac{c_i}{c_n}\right)^2 \omega_i^2}{c_n^2 + c_n^2 \sum_{i=1}^{n-1} \left(\frac{c_i}{c_n}\right)^2} \qquad (E_1)$$

Let $\left|\frac{c_i}{c_n}\right| = \epsilon_i << 1$. Then Eq. (E_1) becomes

$$R(\vec{x}) = \frac{\omega_n^2 + \sum_{i=1}^{n-1} \epsilon_i^2 \omega_i^2}{1 + \sum_{i=1}^{n-1} \epsilon_i^2} \simeq \left(\omega_n^2 + \sum_{i=1}^{n-1} \epsilon_i^2 \omega_i^2\right)\left(1 - \sum_{i=1}^{n-1} \epsilon_i^2\right)$$

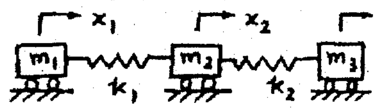
$$\simeq \omega_n^2 + \sum_{i=1}^{n-1} \epsilon_i^2 \omega_i^2 - \omega_n^2 \sum_{i=1}^{n-1} \epsilon_i^2$$

$$\simeq \omega_n^2 + \sum_{i=1}^{n-1} (\omega_i^2 - \omega_n^2)\, \epsilon_i^2 \qquad (E_2)$$

Since $\omega_i^2 < \omega_n^2$, in general, (E_2) shows that

$$R(\vec{x}) \le \omega_n^2$$

(7.13) Equations of motion



$$m_1 \ddot{x}_1 + k_1 (x_1 - x_2) = 0 \qquad \text{--- } (E.1)$$

$$m_2 \ddot{x}_2 + k_1 (x_2 - x_1) + k_2 (x_2 - x_3) = 0 \qquad \text{--- } (E.2)$$

$$m_3 \ddot{x}_3 + k_2 (x_3 - x_2) = 0 \qquad \text{--- } (E.3)$$

With $x_i(t) = X_i \cos \omega t$, Eqs. (E.1) and (E.2) give

$$m_1 \omega^2 X_1 = k_1 (X_1 - X_2) \quad ; \quad m_2 \omega^2 X_2 = -\omega^2 m_1 X_1 + k_2 (X_2 - X_3)$$

or $\quad X_2 = X_1 - \dfrac{\omega^2 m_1 X_1}{k_1} \quad ; \quad X_3 = X_2 - \dfrac{\omega^2}{k_2} (m_1 X_1 + m_2 X_2)$

Since the system is free-free, the resultant force applied to mass 3,

$$F = \sum_{i=1}^{3} \omega^2 m_i X_i \ , \quad \text{must be zero.}$$

The computer program, with the subroutine FUN and the output, are given below.

```
C =====================================================================
C
C HOLZER METHOD
C THIS PROGRAM REQUIRES USER SUPPLIED SUBROUTINE FUN
C
C =====================================================================
C NT = NUMBER OF TIMES INCREMENT OF OM CHANGED
C FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
C NR = NUMBER OF ROOTS REQUIRED
      NR=2    <-----
C END OF PROBLEM-DEPENDENT DATA
      NFUN=1
      OM=0.1 <-----
      CALL FUN (OM,F)
      PRINT 60,NFUN,OM,F
      FB=F
      INR=0
      OM=0.0
 100  DEL=10.0 <-----
      NT=0
 200  CONTINUE
      F1=FB
 10   OM=OM+DEL
      CALL FUN (OM,F)
      NFUN=NFUN+1
      PRINT 60,NFUN,OM,F
      F2=F1*F
      IF (F2 .LT. 0.0) GO TO 20
      F1=F
      GO TO 10
 20   NT=NT+1
      PRINT 30, OM,F,DEL,NFUN
 30   FORMAT (//,31H CHANGE OF SIGN DETECTED AT OM=,E15.8,/,3H F=,
     2 E15.8,/,5H DEL=,E15.8,/,6H NFUN=,I5,/)
      IF (NT .EQ. 6) GO TO 40
      IF (NT .EQ. 1) OMB=OM
      OM=OM-DEL
      DEL=DEL/10.0
      GO TO 200
 40   INR=INR+1
      IF (INR .EQ. NR) GO TO 50
      OM=OMB
      CALL FUN (OM,F)
      NFUN=NFUN+1
      PRINT 60,NFUN,OM,F
      FB=F
      GO TO 100
 50   CONTINUE
 60   FORMAT (2X,6H NFUN=,I4,2X,4H OM=,E15.8,2X,3H F=,E15.8)
      STOP
      END
C =====================================================================
C
C SUBROUTINE FUN
C
C =====================================================================
```

```
      SUBROUTINE FUN (OM,F)
      XM1=100.0
      XM2=20.0
      XM3=200.0
      XK1=8000.0
      XK2=4000.0
      OMS=OM**2
      X1=1.0
      X2=(1.0-(OMS*XM1/XK1))*X1
      X3=X2-(OMS/XK2)*(XM1*X1+XM2*X2)
      F=OMS*(XM1*X1+XM2*X2+XM3*X3)
      RETURN
      END
```

$\omega_1 = 0$

```
   NFUN=    1    OM= 0.10000000E+00    F= 0.31991253E+01
   NFUN=    2    OM= 0.10000000E+02    F=-0.43000000E+05

 CHANGE OF SIGN DETECTED AT OM= 0.10000000E+02
 F=-0.43000000E+05
 DEL= 0.10000000E+02
 NFUN=    2

   NFUN=    3    OM= 0.10000000E+01    F= 0.31126251E+03
   NFUN=    4    OM= 0.20000000E+01    F= 0.11408000E+04
   NFUN=    5    OM= 0.30000000E+01    F= 0.21803625E+04
   NFUN=    6    OM= 0.40000000E+01    F= 0.29312000E+04
   NFUN=    7    OM= 0.50000000E+01    F= 0.27265625E+04
   NFUN=    8    OM= 0.60000000E+01    F= 0.76320044E+03
   NFUN=    9    OM= 0.70000000E+01    F=-0.38581377E+04

 CHANGE OF SIGN DETECTED AT OM= 0.70000000E+01
 F=-0.38581377E+04
 DEL= 0.10000000E+01
 NFUN=    9
    :
 CHANGE OF SIGN DETECTED AT OM= 0.62220016E+01
 F=-0.28649828E+00
 DEL= 0.99999990E-01
 NFUN=   27

   NFUN=   28    OM= 0.10000000E+02    F=-0.43000000E+05
   NFUN=   29    OM= 0.20000000E+02    F=-0.47200000E+06
   NFUN=   30    OM= 0.30000000E+02    F= 0.23130000E+07

 CHANGE OF SIGN DETECTED AT OM= 0.30000000E+02
 F= 0.23130000E+07
 DEL= 0.10000000E+02
 NFUN=   30

   NFUN=   31    OM= 0.21000000E+02    F=-0.18851225E+06
   NFUN=   32    OM= 0.22000000E+02    F=-0.47761113E+06
   NFUN=   33    OM= 0.23000000E+02    F=-0.42888006E+06
```

$\omega_2 = 6.2220016$

⋮

```
CHANGE OF SIGN DETECTED AT OM= 0.25715595E+02
F= 0.21467506E+02
DEL= 0.99999990E-04
NFUN=    58
```

← $\omega_3 = 25.715595$

(7.14) Eigenvalue problem is

$$\begin{bmatrix} (\lambda-2) & 1 & 0 \\ 1 & (\lambda-2) & 1 \\ 0 & 1 & (2\lambda-3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (E_1)$$

where $\lambda = \frac{m\omega^2}{k}$.

If $x_1 = 1$, (E_1) gives $x_2 = -(\lambda-2)\,x_1$, $x_3 = -x_1 - (\lambda-2)\,x_2$,

$E = x_2 + (2\lambda-3)\,x_3$ (should be zero)

The computer program and results are given.

```
C ========================================================================
C
C HOLZER METHOD
C THIS PROGRAM REQUIRES USER SUPPLIED SUBROUTINE FUN
C
C ========================================================================
C NT = NUMBER OF TIMES INCREMENT OF OM CHANGED
C FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
C NR = NUMBER OF ROOTS REQUIRED
      NR=3 <-----
C END OF PROBLEM-DEPENDENT DATA
      NFUN=1
      OM=0.01 <-----
      CALL FUN (OM,F)
      WRITE(13,60) NFUN,OM,F
      FB=F
      INR=0
      OM=0.0
 100  DEL=0.25 <-----
      NT=0
 200  CONTINUE
      F1=FB
 10   OM=OM+DEL
      CALL FUN (OM,F)
      NFUN=NFUN+1
      WRITE(13,60) NFUN,OM,F
      F2=F1*F
      IF (F2 .LT. 0.0) GO TO 20
      F1=F
      GO TO 10
 20   NT=NT+1
      WRITE(13,30) OM,F,DEL,NFUN
 30   FORMAT (//,31H CHANGE OF SIGN DETECTED AT OM=,E15.8,/,3H F=,
     2 E15.8,/,5H DEL=,E15.8,/,6H NFUN=,I5,/)
      IF (NT .EQ. 6) GO TO 40
      IF (NT .EQ. 1) OMB=OM
      OM=OM-DEL
      DEL=DEL/10.0
      GO TO 200
 40   INR=INR+1
```

```
      IF (INR .EQ. NR) GO TO 50
      OM=OMB
      CALL FUN (OM,F)
      NFUN=NFUN+1
      WRITE(13,60) NFUN,OM,F
      FB=F
      GO TO 100
 50   CONTINUE
 60   FORMAT (2X,6H NFUN=,I4,2X,4H OM=,E15.8,2X,3H F=,E15.8)
      STOP
      END
C ===========================================================================
C
C SUBROUTINE FUN
C
C ===========================================================================
      SUBROUTINE FUN (OM,F)
      X1=1.0
      X2=-(OM-2.0)*X1
      X3=-X1-(OM-2.0)*X2
      F=X2+(2.0*OM-3.0)*X3
      RETURN
      END
```

```
  NFUN=    1    OM= 0.99999998E-02    F=-0.68310986E+01
  NFUN=    2    OM= 0.25000000E+00    F=-0.34062500E+01
  NFUN=    3    OM= 0.50000000E+00    F=-0.10000000E+01
  NFUN=    4    OM= 0.75000000E+00    F= 0.40625000E+00


CHANGE OF SIGN DETECTED AT OM= 0.75000000E+00
F= 0.40625000E+00
DEL= 0.25000000E+00
NFUN=    4

  NFUN=    5    OM= 0.52499998E+00    F=-0.81746900E+00
  NFUN=    6    OM= 0.54999995E+00    F=-0.64475036E+00
  NFUN=    7    OM= 0.57499993E+00    F=-0.48165655E+00
  NFUN=    8    OM= 0.59999990E+00    F=-0.32800055E+00
  NFUN=    9    OM= 0.62499988E+00    F=-0.18359447E+00
  NFUN=   10    OM= 0.64999986E+00    F=-0.48250675E-01
  NFUN=   11    OM= 0.67499983E+00    F= 0.78217864E-01
   :
  NFUN=   28    OM= 0.65932715E+00    F=-0.38385391E-04
  NFUN=   29    OM= 0.65932965E+00    F=-0.25749207E-04
  NFUN=   30    OM= 0.65933216E+00    F=-0.12755394E-04
  NFUN=   31    OM= 0.65933466E+00    F=-0.11920929E-06
  NFUN=   32    OM= 0.65933716E+00    F= 0.12636185E-04


CHANGE OF SIGN DETECTED AT OM= 0.65933716E+00      ← λ1 = 0·65933716
F= 0.12636185E-04
DEL= 0.24999997E-05
NFUN=   32

  NFUN=   33    OM= 0.75000000E+00    F= 0.40625000E+00
```

```
NFUN=  34   OM= 0.10000000E+01   F= 0.10000000E+01
NFUN=  35   OM= 0.12500000E+01   F= 0.96875000E+00
NFUN=  36   OM= 0.15000000E+01   F= 0.50000000E+00
NFUN=  37   OM= 0.17500000E+01   F=-0.21875000E+00
 :
NFUN=  63   OM= 0.16789526E+01   F= 0.32067299E-04
NFUN=  64   OM= 0.16789551E+01   F= 0.24497509E-04
NFUN=  65   OM= 0.16789576E+01   F= 0.16927719E-04
NFUN=  66   OM= 0.16789601E+01   F= 0.93579292E-05
NFUN=  67   OM= 0.16789626E+01   F= 0.17881393E-05
NFUN=  68   OM= 0.16789651E+01   F=-0.57816505E-05

CHANGE OF SIGN DETECTED AT OM= 0.16789651E+01
F=-0.57816505E-05
DEL= 0.24999997E-05
NFUN=   68
```

$\leftarrow \lambda_2 = 1{\cdot}6789651$

```
NFUN=  69   OM= 0.17500000E+01   F=-0.21875000E+00
NFUN=  70   OM= 0.20000000E+01   F=-0.10000000E+01
NFUN=  71   OM= 0.22500000E+01   F=-0.16562500E+01
NFUN=  72   OM= 0.25000000E+01   F=-0.20000000E+01
NFUN=  73   OM= 0.27500000E+01   F=-0.18437500E+01
NFUN=  74   OM= 0.30000000E+01   F=-0.10000000E+01
NFUN=  75   OM= 0.32500000E+01   F= 0.71875000E+00
 :
NFUN=  95   OM= 0.31615264E+01   F=-0.13036728E-02
NFUN=  96   OM= 0.31615515E+01   F=-0.11179447E-02
NFUN=  97   OM= 0.31615765E+01   F=-0.93221664E-03
NFUN=  98   OM= 0.31616015E+01   F=-0.74636936E-03
NFUN=  99   OM= 0.31616266E+01   F=-0.56064129E-03
NFUN= 100   OM= 0.31616516E+01   F=-0.37479401E-03
NFUN= 101   OM= 0.31616766E+01   F=-0.18906593E-03
NFUN= 102   OM= 0.31617017E+01   F=-0.32186508E-05
NFUN= 103   OM= 0.31617267E+01   F= 0.18215179E-03

CHANGE OF SIGN DETECTED AT OM= 0.31617267E+01
F= 0.18215179E-03
DEL= 0.24999998E-04
NFUN=  103
```

$\leftarrow \lambda_3 = 3{\cdot}1617267$

7.15 The program listed in Problems 7·13 and 7·14 is used with NR = 1, initial value of OM = 0·01, DEL = 0·25 and

```
C =====================================================================
C
C SUBROUTINE FUN
C
C =====================================================================
      SUBROUTINE FUN (OM,F)
      X1=1.0
      X2=(2.0-OM)*X1
      X3=-X1+(2.0-OM)*X2
      F=-X2+(1.0-OM)*X3
```

$\Theta_1 = 1$

$\Theta_2 = (-\lambda+2)\,\Theta_1 \quad ; \lambda = \left(\frac{J_0\omega^2}{k_t}\right)$

$\Theta_3 = -\Theta_1 + (-\lambda+2)\,\Theta_2$

$F = -\Theta_2 + (-\lambda+1)\,\Theta_3$

```
      RETURN
      END
```

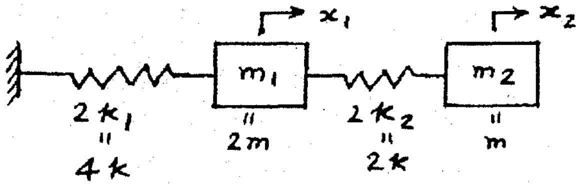
The output of the program is given below.

```
 NFUN=    1    OM= 0.99999998E-02    F= 0.94049907E+00
 NFUN=    2    OM= 0.25000000E+00    F=-0.20312500E+00

CHANGE OF SIGN DETECTED AT OM= 0.25000000E+00
F=-0.20312500E+00
DEL= 0.25000000E+00
NFUN=    2

 NFUN=    3    OM= 0.25000000E-01    F= 0.85310948E+00
 NFUN=    4    OM= 0.50000001E-01    F= 0.71237516E+00
 NFUN=    5    OM= 0.75000003E-01    F= 0.57770300E+00
 NFUN=    6    OM= 0.10000000E+00    F= 0.44899976E+00
 NFUN=    7    OM= 0.12500000E+00    F= 0.32617188E+00
 NFUN=    8    OM= 0.15000001E+00    F= 0.20912516E+00
 NFUN=    9    OM= 0.17500001E+00    F= 0.97765565E-01
 NFUN=   10    OM= 0.20000002E+00    F=-0.80001354E-02

CHANGE OF SIGN DETECTED AT OM= 0.20000002E+00
F=-0.80001354E-02
DEL= 0.25000000E-01
NFUN=   10

 NFUN=   11    OM= 0.17750001E+00    F= 0.86938858E-01
 NFUN=   12    OM= 0.18000001E+00    F= 0.76168060E-01
 NFUN=   13    OM= 0.18250000E+00    F= 0.65452933E-01
 NFUN=   14    OM= 0.18500000E+00    F= 0.54793477E-01
 NFUN=   15    OM= 0.18750000E+00    F= 0.44189453E-01
 NFUN=   16    OM= 0.19000000E+00    F= 0.33641100E-01
 NFUN=   17    OM= 0.19250000E+00    F= 0.23147941E-01
 NFUN=   18    OM= 0.19499999E+00    F= 0.12710214E-01
 NFUN=   19    OM= 0.19749999E+00    F= 0.23275614E-02
 NFUN=   20    OM= 0.19999999E+00    F=-0.79998970E-02

CHANGE OF SIGN DETECTED AT OM= 0.19999999E+00
F=-0.79998970E-02
DEL= 0.24999999E-02
NFUN=   20

 NFUN=   21    OM= 0.19774999E+00    F= 0.12923479E-02
 NFUN=   22    OM= 0.19799998E+00    F= 0.25773048E-03
 NFUN=   23    OM= 0.19824998E+00    F=-0.77641010E-03

CHANGE OF SIGN DETECTED AT OM= 0.19824998E+00
F=-0.77641010E-03
DEL= 0.24999998E-03
NFUN=   23
```

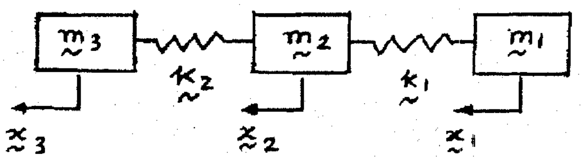
```
NFUN=  24   OM= 0.19802499E+00    F= 0.15425682E-03
NFUN=  25   OM= 0.19804999E+00    F= 0.50902367E-04
NFUN=  26   OM= 0.19807500E+00    F=-0.52571297E-04

CHANGE OF SIGN DETECTED AT OM= 0.19807500E+00
F=-0.52571297E-04
DEL= 0.24999998E-04
NFUN=   26

NFUN=  27   OM= 0.19805250E+00    F= 0.40531158E-04
NFUN=  28   OM= 0.19805500E+00    F= 0.30040741E-04
NFUN=  29   OM= 0.19805750E+00    F= 0.19669533E-04
NFUN=  30   OM= 0.19806001E+00    F= 0.92983246E-05
NFUN=  31   OM= 0.19806251E+00    F=-0.10728836E-05

CHANGE OF SIGN DETECTED AT OM= 0.19806251E+00
F=-0.10728836E-05
DEL= 0.24999997E-05
NFUN=   31
```

← $\lambda_1 = 0.19806251$

7.16 The system can be modeled as shown.



The system can be redrawn as follows:



Here $\underset{\sim}{m}_1 = m$, $\underset{\sim}{k}_1 = 2k$, $\underset{\sim}{m}_2 = 2m$, $\underset{\sim}{k}_2 = 4k$, $\underset{\sim}{x}_1 = x_2$, $\underset{\sim}{x}_2 = x_1$, $\underset{\sim}{x}_3 = 0$, $\underset{\sim}{m}_3$ = any value.

Let $\underset{\sim}{X}_1 = 1$

$$\underset{\sim}{X}_2 = \underset{\sim}{X}_1 - \frac{\omega^2 \underset{\sim}{m}_1 \underset{\sim}{X}_1}{\underset{\sim}{k}_1}$$

$$= \underset{\sim}{X}_1 \left(1 - \frac{m\omega^2}{2k}\right)$$

$$\underset{\sim}{X}_3 = \underset{\sim}{X}_2 - \frac{\omega^2}{\underset{\sim}{k}_2}\left(\underset{\sim}{m}_1 \underset{\sim}{X}_1 + \underset{\sim}{m}_2 \underset{\sim}{X}_2\right)$$

$$= \underset{\sim}{X}_2 \left(1 - \frac{m\omega^2}{2k}\right) - \underset{\sim}{X}_1 \left(\frac{m\omega^2}{4k}\right)$$

$$= \underset{\sim}{X}_1 \left(1 + \frac{m^2\omega^4}{4k^2} - \frac{5m\omega^2}{4k}\right)$$

Holzer's procedure involves assuming different values for ω^2 and finding, out of those values, the correct frequency ω as the one which gives $\underset{\sim}{X}_3 = 0$ (boundary condition to be satisfied).

From the expression for $\underset{\sim}{X}_3$, we can find the correct frequency (without trial and error) by setting

$$\omega^4 \left(\frac{m^2}{4k^2} \right) - \omega^2 \left(\frac{5m}{4k} \right) + 1 = 0$$

or $$\omega^2 = \frac{\left(\frac{5m}{4k}\right) \pm \sqrt{\frac{25m^2}{16k^2} - \frac{m^2}{k^2}}}{(m^2/2k^2)} = \frac{\frac{5m}{4k} \pm \frac{3m}{4k}}{(m^2/2k^2)}$$

Thus the first natural frequency is given by

$$\omega_1^2 = \left(\frac{5m}{4k} - \frac{3m}{4k}\right) \Big/ \left(\frac{m^2}{2k^2}\right) \quad \text{or} \quad \omega_1 = \sqrt{\frac{k}{m}}\,.$$

7.17 Eqs. (7.32) to (7.35) give

$$\Theta_2 = \Theta_1 - \frac{\omega^2 J_1}{k_{t1}} \Theta_1$$

$$\Theta_3 = \Theta_2 - \frac{\omega^2}{k_{t2}} (J_1 \Theta_1 + J_2 \Theta_2)$$

$$E = \sum_{i=1}^{3} \omega^2 J_i \Theta_i = \text{sum of inertia torques (should be zero)}$$

The computer program of Problems 7.13 and 7.14 is used with NR = 3, OM = 0.01 and DEL = 10.0.

The subroutine FUN and results are given.

```
C ==========================================================================
C
C SUBROUTINE FUN
C
C ==========================================================================
      SUBROUTINE FUN (OM,F)
      XJ1=10.0
      XJ2=5.0
      XJ3=1.0
      XKT1=1.0E+06
      XKT2=1.0E+06
      X1=1.0
      OMS=OM**2
      X2=X1-(OMS*XJ1/XKT1)*X1
      X3=X2-(OMS/XKT2)*(XJ1*X1+XJ2*X2)
      F=OMS*(XJ1*X1+XJ2*X2+XJ3*X3)
      RETURN
      END
```

$\omega_1 = 0$

```
  NFUN=   1    OM= 0.99999998E-02    F= 0.16000000E-02
  NFUN=   2    OM= 0.10000000E+02    F= 0.15992500E+04
  NFUN=   3    OM= 0.20000000E+02    F= 0.63880034E+04
  NFUN=   4    OM= 0.30000000E+02    F= 0.14339287E+05
  NFUN=   5    OM= 0.40000000E+02    F= 0.25408205E+05
  NFUN=   6    OM= 0.50000000E+02    F= 0.39532031E+05
  NFUN=   7    OM= 0.60000000E+02    F= 0.56630336E+05
  NFUN=   8    OM= 0.70000000E+02    F= 0.76605133E+05
  NFUN=   9    OM= 0.80000000E+02    F= 0.99341109E+05
  NFUN=  10    OM= 0.90000000E+02    F= 0.12470582E+06
  :
```

```
  NFUN=  50   OM= 0.49000000E+03   F= 0.21006358E+06
  NFUN=  51   OM= 0.50000000E+03   F= 0.93750000E+05
  NFUN=  52   OM= 0.51000000E+03   F=-0.32486393E+05

CHANGE OF SIGN DETECTED AT OM= 0.51000000E+03
F=-0.32486393E+05
DEL= 0.10000000E+02
NFUN=   52
  :
  NFUN=  69   OM= 0.50750012E+03   F= 0.93337460E+01
  NFUN=  70   OM= 0.50750021E+03   F= 0.79214058E+01
  NFUN=  71   OM= 0.50750031E+03   F= 0.66932836E+01
  NFUN=  72   OM= 0.50750040E+03   F= 0.57721915E+01
  NFUN=  73   OM= 0.50750049E+03   F= 0.45440674E+01
  NFUN=  74   OM= 0.50750058E+03   F= 0.33159423E+01
  NFUN=  75   OM= 0.50750067E+03   F= 0.20878162E+01
  NFUN=  76   OM= 0.50750076E+03   F= 0.92109573E+00
  NFUN=  77   OM= 0.50750085E+03   F=-0.30703202E+00

CHANGE OF SIGN DETECTED AT OM= 0.50750085E+03
F=-0.30703202E+00
DEL= 0.99999990E-04
NFUN=   77

  NFUN=  78   OM= 0.51000000E+03   F=-0.32486393E+05
  NFUN=  79   OM= 0.52000000E+03   F=-0.16878158E+06
  NFUN=  80   OM= 0.53000000E+03   F=-0.31524272E+06
  :
  NFUN= 137   OM= 0.11000000E+04   F=-0.18694431E+07
  NFUN= 138   OM= 0.11100000E+04   F=-0.62095219E+06
  NFUN= 139   OM= 0.11200000E+04   F= 0.74758494E+06

CHANGE OF SIGN DETECTED AT OM= 0.11200000E+04
F= 0.74758494E+06
DEL= 0.10000000E+02
NFUN=  139
  :
  NFUN= 168   OM= 0.11146489E+04   F=-0.37916328E+02
  NFUN= 169   OM= 0.11146490E+04   F=-0.23697710E+02
  NFUN= 170   OM= 0.11146492E+04   F=-0.94790859E+01
  NFUN= 171   OM= 0.11146493E+04   F= 0.47395439E+01

CHANGE OF SIGN DETECTED AT OM= 0.11146493E+04
F= 0.47395439E+01
DEL= 0.99999990E-04
NFUN=  171
```

← $\omega_2 = 507.50085$

← $\omega_3 = 1114.6493$

7.18 Eigenvector $\vec{X}^{(1)}$ corresponding to $\lambda_1 = 10.38068 \left(= \frac{1}{\omega_1^2}\right)$ is given by

$$\begin{bmatrix} (2.5-\lambda_1) & -1 & 0 \\ -1 & (5-\lambda_1) & -\sqrt{2} \\ 0 & -\sqrt{2} & (10-\lambda_1) \end{bmatrix} \begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

i.e. $x_2^{(1)} = -7.88068\ x_1^{(1)}$ and $x_3^{(1)} = -3.71497\ x_2^{(1)} = 29.27649\ x_1^{(1)}$

$$\vec{X}^{(1)} = \alpha \begin{Bmatrix} 1 \\ -7.88068 \\ 29.27649 \end{Bmatrix} \quad \text{where } \alpha = \text{value of } x_1^{(1)}$$

when $\vec{X}^{(1)}$ is normalized as $\vec{X}^{(1)T}[m]\vec{X}^{(1)} = 1$, $\alpha = 0.03296$

$$\therefore \vec{X}^{(1)} = \begin{Bmatrix} 0.03296 \\ -0.25975 \\ 0.96495 \end{Bmatrix}$$

$$[D_2] = [D_1] - \lambda_1 \vec{X}^{(1)} \vec{X}^{(1)T} [m]$$

$$= \begin{bmatrix} 2.5 & -1.0 & 0.0 \\ -1.0 & 5.0 & -1.4142 \\ 0.0 & -1.4142 & 10.0 \end{bmatrix} - (10.38068) \begin{Bmatrix} 0.03296 \\ -0.25975 \\ 0.96495 \end{Bmatrix} \{0.03296 \mid -0.25975 \mid 0.96495\}$$

$$= \begin{bmatrix} 2.4887228 & -0.9111273 & -0.3301549 \\ -0.9111273 & 4.2996149 & 1.1876738 \\ -0.3301549 & 1.1876738 & 0.3342530 \end{bmatrix}$$

If $\vec{X}_0 = \begin{Bmatrix} 1.0 \\ -7.88068 \\ 29.27649 \end{Bmatrix}$ is used, the iterative procedure $\vec{X}_{i+1} = [D_2]\vec{X}_i$ gives the following results (with $X_{1,i+1} = 1$):

Iter. No. (i)	$\vec{X}_{i+1}$ as a row vector (with $X_{1,i+1} = 1$)
ITER= 0	0.10000000E+01-0.78806801E+01 0.29276489E+02
ITER= 1	0.10000000E+01-0.10666667E+02 0.29333334E+02
ITER= 2	0.10000000E+01-0.47272644E+01-0.13066854E+01
ITER= 3	0.10000000E+01-0.31531227E+01-0.88291872E+00
ITER= 4	0.10000000E+01-0.27448514E+01-0.77301961E+00
ITER= 5	0.10000000E+01-0.25989382E+01-0.73374254E+00
ITER= 6	0.10000000E+01-0.25411220E+01-0.71817952E+00
ITER= 7	0.10000000E+01-0.25172873E+01-0.71176374E+00
ITER= 8	0.10000000E+01-0.25073018E+01-0.70907575E+00
ITER= 9	0.10000000E+01-0.25030899E+01-0.70794201E+00
ITER= 10	0.10000000E+01-0.25013082E+01-0.70746231E+00
ITER= 11	0.10000000E+01-0.25005538E+01-0.70725930E+00
ITER= 12	0.10000000E+01-0.25002341E+01-0.70717323E+00
ITER= 13	0.10000000E+01-0.25000987E+01-0.70713681E+00
ITER= 14	0.10000000E+01-0.25000415E+01-0.70712131E+00
ITER= 15	0.10000000E+01-0.25000172E+01-0.70711482E+00

converged value of $\lambda_2 = 5.0004216$ (or $\omega_2 = 0.4472117$)

By using a similar procedure, $[D_3]$ is found. With the same $\vec{X}_0$, the results obtained from $\vec{X}_{i+1} = [D_3]\vec{X}_i$ are given below.

Iter. no. (i)	$\vec{X}_{i+1}$ as a row vector (with $X_{1,i+1}=1$)
ITER= 0	0.10000000E+01 -0.78806801E+01 0.29276489E+02
ITER= 1	0.10000000E+01 0.19600000E+02 -0.72824997E+02
ITER= 2	0.10000000E+01 0.38067123E+00 0.68348452E-01
ITER= 3	0.10000000E+01 0.38067168E+00 0.68312615E-01
ITER= 4	0.10000000E+01 0.38067159E+00 0.68312593E-01

Converged value of $\lambda_3 = 2.11932243$ ($\omega_3 = 0.68691260$).

(7.19) $[k]^{-1} = \frac{1}{k}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 2 & 2.5\end{bmatrix}$, $[D] = [k]^{-1}[m] = \frac{m}{k}\begin{bmatrix}1 & 1 & 2\\ 1 & 2 & 4\\ 1 & 2 & 5\end{bmatrix}$

Using $\vec{X}_0 = \begin{Bmatrix}1\\1\\1\end{Bmatrix}$, the following results are obtained:

Iteration number (i)	$\vec{X}_{i+1} = [D]\vec{X}_i$ with $X_{1,i} = 1$ (given as row vector)
ITER= 0	0.10000000E+01 0.10000000E+01 0.10000000E+01
ITER= 1	0.10000000E+01 0.17500000E+01 0.20000000E+01
ITER= 2	0.10000000E+01 0.18518518E+01 0.21481481E+01
ITER= 3	0.10000000E+01 0.18601036E+01 0.21606219E+01
ITER= 4	0.10000000E+01 0.18607503E+01 0.21616161E+01
ITER= 5	0.10000000E+01 0.18608015E+01 0.21616952E+01
ITER= 6	0.10000000E+01 0.18608055E+01 0.21617017E+01 ← $\vec{X}^{(1)}$

converged frequency $= 0.373088 = \omega_1\sqrt{\frac{m}{k}}$

Repetition of the procedure with $[D_2]$ and $[D_3]$ gives the following results:

$$\omega_2\sqrt{\frac{m}{k}} = 1.321324$$

$$\vec{X}^{(2)} = \{1.000000 \quad 0.254098 \quad -0.340706\}$$

$$\omega_3\sqrt{\frac{m}{k}} = 2.028523$$

$$\vec{X}^{(3)} = \{1.000000 \quad -2.114801 \quad 0.678847\}$$

(7.20) $[k]^{-1} = \frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.5 \\ 1 & 1.5 & 1.833 \end{bmatrix}$, $[k]^{-1}[m] = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.5 \\ 1 & 1.5 & 1.833 \end{bmatrix}$

$$[D] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.5 \\ 1 & 1.5 & 1.833 \end{bmatrix}$$

With $\vec{X}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$, the following results are obtained (Program 14):

Iteration number (i)	$\vec{X}_{i+1} = [D]\,\vec{X}_i$ with $x_{1,i} = 1$ (given as row vector)		
0	0.10000000E+01	0.10000000E+01	0.10000000E+01
1	0.10000000E+01	0.13333334E+01	0.14443334E+01
2	0.10000000E+01	0.13676430E+01	0.14949605E+01
3	0.10000000E+01	0.13705537E+01	0.14994360E+01
4	0.10000000E+01	0.13708007E+01	0.14998223E+01
5	0.10000000E+01	0.13708220E+01	0.14998555E+01
6	0.10000000E+01	0.13708237E+01	0.14998584E+01 ← $\vec{X}^{(1)}$

converged frequency = 0.50828409 = ω_1

0	0.10000000E+01	0.10000000E+01	0.10000000E+01
1	0.10000000E+01	-0.56849640E-01	-0.61459929E+00
2	0.10000000E+01	-0.22656711E-01	-0.64602280E+00
3	0.10000000E+01	-0.91092410E-02	-0.65840477E+00
4	0.10000000E+01	-0.38191630E-02	-0.66323972E+00
5	0.10000000E+01	-0.17635337E-02	-0.66511846E+00
6	0.10000000E+01	-0.96627331E-03	-0.66584712E+00
7	0.10000000E+01	-0.65728393E-03	-0.66612953E+00
8	0.10000000E+01	-0.53756207E-03	-0.66623896E+00
9	0.10000000E+01	-0.49118389E-03	-0.66628140E+00
10	0.10000000E+01	-0.47320157E-03	-0.66629785E+00
11	0.10000000E+01	-0.46624581E-03	-0.66630417E+00
12	0.10000000E+01	-0.46355074E-03	-0.66630661E+00
13	0.10000000E+01	-0.46252197E-03	-0.66630769E+00
14	0.10000000E+01	-0.46209697E-03	-0.66630799E+00
15	0.10000000E+01	-0.46194042E-03	-0.66630810E+00
16	0.10000000E+01	-0.46188448E-03	-0.66630810E+00
17	0.10000000E+01	-0.46187337E-03	-0.66630816E+00
18	0.10000000E+01	-0.46186216E-03	-0.66630822E+00
19	0.10000000E+01	-0.46185096E-03	-0.66630816E+00
20	0.10000000E+01	-0.46185098E-03	-0.66630822E+00 ← $\vec{X}^{(2)}$

converged frequency = 1.7323176 = ω_2

0	0.10000000E+01	0.10000000E+01	0.10000000E+01
1	0.10000000E+01	-0.23654659E+01	0.15024464E+01
2	0.10000000E+01	-0.23733659E+01	0.15024524E+01
3	0.10000000E+01	-0.23733664E+01	0.15024524E+01 ← $\vec{X}^{(3)}$

converged frequency = 2.783294 = ω_3

(7.21) From problem 6.18, $[k] = \frac{GJ}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $[J_d] = J_0\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$[D] = \begin{bmatrix} 0.75 & 0.50 & 0.25 \\ 0.50 & 1.00 & 0.50 \\ 0.25 & 0.50 & 0.75 \end{bmatrix}$$

With $\vec{X}_0 = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$, the following results are obtained (Program 14):

Iteration number (i)	$\vec{X}_{i+1} = [D]\vec{X}_i$ with $x_{1,i} = 1$ (given as row vector)			
0	0.10000000E+01	0.20000000E+01	0.30000000E+01	
1	0.10000000E+01	0.16000000E+01	0.14000000E+01	
2	0.10000000E+01	0.14736842E+01	0.11052632E+01	
3	0.10000000E+01	0.14328358E+01	0.10298507E+01	
4	0.10000000E+01	0.14199134E+01	0.10086581E+01	
5	0.10000000E+01	0.14159292E+01	0.10025285E+01	
6	0.10000000E+01	0.14147242E+01	0.10007399E+01	
7	0.10000000E+01	0.14143645E+01	0.10002166E+01	
8	0.10000000E+01	0.14142580E+01	0.10000634E+01	
9	0.10000000E+01	0.14142267E+01	0.10000186E+01	
10	0.10000000E+01	0.14142175E+01	0.10000055E+01	
11	0.10000000E+01	0.14142147E+01	0.10000015E+01	$\leftarrow \vec{X}^{(1)}$

converged frequency = 0.76536608 = ω_1

0	0.10000000E+01	0.20000000E+01	0.30000000E+01	
1	0.10000000E+01	0.29291141E+00	-0.14141805E+01	
2	0.10000000E+01	0.15801974E+00	-0.12234681E+01	
3	0.10000000E+01	0.88471927E-01	-0.11251128E+01	
4	0.10000000E+01	0.50517395E-01	-0.10714371E+01	
5	0.10000000E+01	0.29161688E-01	-0.10412357E+01	
⋮				
27	0.10000000E+01	0.19669531E-05	-0.99999768E+00	
28	0.10000000E+01	0.18924472E-05	-0.99999750E+00	
29	0.10000000E+01	0.18179417E-05	-0.99999750E+00	
30	0.10000000E+01	0.18030403E-05	-0.99999738E+00	
31	0.10000000E+01	0.17732382E-05	-0.99999744E+00	
32	0.10000000E+01	0.17732380E-05	-0.99999726E+00	$\leftarrow \vec{X}^{(2)}$

converged frequency = 1.4142134 = ω_2

0	0.10000000E+01	0.20000000E+01	0.30000000E+01	
1	0.10000000E+01	-0.14143940E+01	0.10000018E+01	
2	0.10000000E+01	-0.14142135E+01	0.10000004E+01	
3	0.10000000E+01	-0.14142135E+01	0.10000004E+01	$\leftarrow \vec{X}^{(3)}$

converged frequency = 1.8477590 = ω_3

7.22 From Problem 7.9, $[k] = \frac{T}{l}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $[m] = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $[k]^{-1} = \frac{l}{3T}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$[D] = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

With $\vec{X}_0 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$, the following results are obtained (Program 14):

Iteration number (i)	$\vec{X}_{i+1} = [D]\vec{X}_i$ with $x_{1,i} = 1$ (given as row vector)	
0	0.10000000E+01	0.20000000E+01
1	0.10000000E+01	0.12500000E+01
2	0.10000000E+01	0.10769231E+01
3	0.10000000E+01	0.10250001E+01
4	0.10000000E+01	0.10082645E+01
5	0.10000000E+01	0.10027474E+01
6	0.10000000E+01	0.10009151E+01
7	0.10000000E+01	0.10003049E+01

8	0.10000000E+01	0.10001017E+01
9	0.10000000E+01	0.10000340E+01
10	0.10000000E+01	0.10000113E+01
11	0.10000000E+01	0.10000038E+01 ← $\vec{x}^{(1)}$

Converged frequency = 0.99999809 = $\omega_1\sqrt{ml/T}$

0	0.10000000E+01	0.20000000E+01
1	0.10000000E+01	-0.99991989E+00
2	0.10000000E+01	-0.99998856E+00
3	0.10000000E+01	-0.99998856E+00 ← $\vec{x}^{(2)}$

Converged frequency = 1.7320508 = $\omega_2\sqrt{ml/T}$

7.23 $[k]^{-1} = \frac{1}{k}\begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1.0 & 1.0 & 1.0 \\ 0.5 & 1.0 & 2.0 & 2.0 \\ 0.5 & 1.0 & 2.0 & 3.0 \end{bmatrix}$, $[D] = \frac{m}{k}\begin{bmatrix} 1.5 & 1.0 & 0.5 & 0.5 \\ 1.5 & 2.0 & 1.0 & 1.0 \\ 1.5 & 2.0 & 2.0 & 2.0 \\ 1.5 & 2.0 & 2.0 & 3.0 \end{bmatrix}$

With $\vec{x}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$, the following results are obtained for $\vec{x}^{(1)}$ and ω_1.

Iteration number (i)	$\vec{x}_{i+1} = [D]\,\vec{x}_i$ with $x_{1,i} = 1$ (given as row vector)			
0	0.10000000E+01	0.10000000E+01	0.10000000E+01	0.10000000E+01
1	0.10000000E+01	0.15714285E+01	0.21428571E+01	0.24285715E+01
2	0.10000000E+01	0.17200000E+01	0.25733335E+01	0.30266669E+01
3	0.10000000E+01	0.17508308E+01	0.26810634E+01	0.31838319E+01
4	0.10000000E+01	0.17574103E+01	0.27059193E+01	0.32208292E+01
5	0.10000000E+01	0.17588729E+01	0.27116063E+01	0.32293591E+01
6	0.10000000E+01	0.17592046E+01	0.27129092E+01	0.32313194E+01
7	0.10000000E+01	0.17592804E+01	0.27132087E+01	0.32317696E+01
8	0.10000000E+01	0.17592978E+01	0.27132771E+01	0.32318730E+01
9	0.10000000E+01	0.17593019E+01	0.27132931E+01	0.32318966E+01

Converged frequency = 0.40058133 = $\omega_1\sqrt{m/k}$

7.24 The problem-dependent data for Program 13 and output are given.

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION D(3,3),E(3,3)
      DATA N,ITMAX,EPS/3,200,1.0E-05/
      DATA D/3.0,-2.0,0.0,-2.0,5.0,-3.0,0.0,-3.0,3.0/
C END OF PROBLEM-DEPENDENT DATA
```

```
EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX
  0.300000E+01   -0.200000E+01    0.000000E+00
 -0.200000E+01    0.500000E+01   -0.300000E+01
  0.000000E+00   -0.300000E+01    0.300000E+01

 EIGEN VALUES ARE
  0.774166E+01    0.300000E+01    0.258343E+00

 EIGEN VECTORS
      FIRST           SECOND          THIRD
   0.335734E+00    0.832048E+00    0.441564E+00
  -0.796032E+00   -0.343903E-05    0.605254E+00
   0.503602E+00   -0.554704E+00    0.662336E+00
```

7.25 The problem-dependent data for Program 13 and output are given.

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
        DIMENSION D(3,3),E(3,3)
        DATA N,ITMAX,EPS/3,200,1.0E-05/
        DATA D/3.0,2.0,1.0,2.0,2.0,1.0,1.0,1.0,1.0/
C END OF PROBLEM-DEPENDENT DATA
```

```
EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX
  0.300000E+01   0.200000E+01   0.100000E+01
  0.200000E+01   0.200000E+01   0.100000E+01
  0.100000E+01   0.100000E+01   0.100000E+01

 EIGEN VALUES ARE
  0.504892E+01   0.643104E+00   0.307979E+00

 EIGEN VECTORS
     FIRST          SECOND          THIRD
   0.736978E+00  -0.591004E+00   0.327991E+00
   0.591007E+00   0.327977E+00  -0.736982E+00
   0.327985E+00   0.736984E+00   0.590999E+00
```

7.26 The main program which calls DECOMP and results are given.

```
C =====================================================================
C
C PROGRAM 15
C MAIN PROGRAM WHICH CALLS DECOMP
C
C =====================================================================
      DIMENSION A(4,4),U(4,4)
      DATA A/4.0,-2.0,6.0,4.0,-2.0,2.0,-1.0,3.0,6.0,-1.0,22.0,13.0,
    2 4.0,3.0,13.0,46.0/
      N=4
      CALL DECOMP (A,U,N)
      WRITE (17,10)
10    FORMAT (/,25H UPPER TRIANGULAR MATRIX:,/)
      DO 30 I=1,N
      WRITE (17,20) (U(I,J),J=1,N)
20    FORMAT (3E15.8)
30    CONTINUE
      STOP
      END
```

```
UPPER TRIANGULAR MATRIX:

0.20000000E+01-0.10000000E+01 0.30000000E+01
0.20000000E+01
0.00000000E+00 0.10000000E+01 0.20000000E+01
0.00000000E+00
0.00000000E+00 0.00000000E+00 0.30000000E+01
0.23333333E+01
0.00000000E+00 0.00000000E+00 0.00000000E+00
0.60461192E+01
```

(7.27) From Eq. (7.84), $u_{11} = \sqrt{5} = 2.236068$, $u_{12} = \frac{a_{12}}{u_{11}} = \frac{-1}{u_{11}} = -0.44721359$,

$$u_{13} = \frac{a_{13}}{u_{11}} = \frac{1}{u_{11}} = 0.44721359$$

$u_{22} = \left(a_{22} - u_{12}^2\right)^{1/2} = 2.408319$, $u_{23} = \frac{1}{u_{22}}\left(a_{23} - u_{12}u_{13}\right) = -1.5778641$

$u_{33} = \left(a_{33} - u_{13}^2 - u_{23}^2\right)^{1/2} = 0.55708611$

$$[U] = \begin{bmatrix} 2.236068 & -0.44721359 & 0.44721359 \\ 0 & 2.408319 & -1.5778641 \\ 0 & 0 & 0.55708611 \end{bmatrix}$$

$\alpha_{11} = \frac{1}{u_{11}} = 0.44721359$ | $\alpha_{12} = -\frac{1}{u_{11}}\left(u_{12}\alpha_{22}\right) = 0.083045475$

$\alpha_{22} = \frac{1}{u_{22}} = 0.41522738$ | $\alpha_{23} = -\frac{1}{u_{22}}\left(u_{23}\alpha_{33}\right) = 1.1760702$

$\alpha_{33} = \frac{1}{u_{33}} = 1.7950547$ | $\alpha_{13} = -\frac{1}{u_{11}}\left(u_{12}\alpha_{23} + u_{13}\alpha_{33}\right) = -0.12379687$

$$[U]^{-1} = [\alpha_{ij}] = \begin{bmatrix} 0.44721359 & 0.083045475 & -0.12379687 \\ 0 & 0.41522738 & 1.1760702 \\ 0 & 0 & 1.7950547 \end{bmatrix}$$

(7.28) From Eqs. (7.84) and (7.86),

$u_{11} = \sqrt{2} = 1.4142135$, $u_{12} = \frac{a_{12}}{u_{11}} = \frac{5}{u_{11}} = 3.5355339$, $u_{13} = \frac{a_{13}}{u_{11}} = 5.6568542$

$u_{22} = \left(a_{22} - u_{12}^2\right)^{1/2} = 1.8708287$, $u_{23} = \frac{1}{u_{22}}\left(a_{23} - u_{12}u_{13}\right) = 4.2761798$

$u_{33} = \left(a_{33} - u_{13}^2 - u_{23}^2\right)^{1/2} = 1.9272485$

$$[U] = \begin{bmatrix} 1.4142135 & 3.5355339 & 5.6568542 \\ 0 & 1.8708287 & 4.2761798 \\ 0 & 0 & 1.9272485 \end{bmatrix}$$

$\alpha_{11} = \frac{1}{u_{11}} = 0.70710677$ | $\alpha_{12} = -\frac{1}{u_{11}}\left(u_{12}\alpha_{22}\right) = -1.3363062$

$\alpha_{22} = \frac{1}{u_{22}} = 0.53452247$ | $\alpha_{23} = -\frac{1}{u_{22}}\left(u_{23}\alpha_{33}\right) = -1.1859988$

$\alpha_{33} = \frac{1}{u_{33}} = 0.51887447$ | $\alpha_{13} = -\frac{1}{u_{11}}\left(u_{12}\alpha_{23} + u_{13}\alpha_{33}\right) = 0.88949931$

$$[U]^{-1} = [\alpha_{ij}] = \begin{bmatrix} 0.70710677 & -1.3363062 & 0.88949931 \\ 0 & 0.53452247 & -1.1859988 \\ 0 & 0 & 0.51887447 \end{bmatrix}$$

(7.29) The problem-dependent data to be used in Program 13 and results are given.

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION D(4,4),E(4,4)
      DATA N,ITMAX,EPS/4,200,1.0E-05/
      DATA D/4.,-2.,6.,4.,-2.,2.,-1.,3.,6.,-1.,22.,13.,4.,3.,13.,46./
C END OF PROBLEM-DEPENDENT DATA
```

```
EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX
  0.400000E+01   -0.200000E+01    0.600000E+01
  0.400000E+01
 -0.200000E+01    0.200000E+01   -0.100000E+01
  0.300000E+01
  0.600000E+01   -0.100000E+01    0.220000E+02
  0.130000E+02
  0.400000E+01    0.300000E+01    0.130000E+02
  0.460000E+02

 EIGEN VALUES ARE
  0.525424E+02    0.178109E+02    0.346931E+01    0.177413E+00

 EIGEN VECTORS
     FIRST            SECOND           THIRD
   0.123182E+00   -0.274316E+00    0.718788E+00
   0.626834E+00    0.407039E-01    0.167114E+00
  -0.614583E+00    0.769873E+00    0.407591E+00
  -0.851753E+00   -0.316805E+00   -0.895647E-01
   0.903902E+00    0.413933E+00    0.725754E-01
  -0.797051E-01
```

(7.30) $[k] = k \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$, $[m] = m \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Eq. (7.84) gives, for $[A] = [k]$,

$$[U] = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 1.41421 & -0.707107 & 0 \\ 0 & 0 & 1.22474 & -0.816497 \\ 0 & 0 & 0 & 0.577350 \end{bmatrix} \sqrt{k}$$

Eq. (7.86) gives

$$[U]^{-1} = \begin{bmatrix} 0.5 & 0.35355338 & 0.20412414 & 0.28867510 \\ 0 & 0.70710677 & 0.40824828 & 0.57735020 \\ 0 & 0 & 0.81649655 & 1.1547004 \\ 0 & 0 & 0 & 1.7320507 \end{bmatrix} \frac{1}{\sqrt{k}}$$

standard eigenproblem is $\quad [D]\vec{Y} = \lambda \vec{Y}$

where

$$[D] = \left([U]^{-1}\right)^T [m] [U]^{-1}$$

$$= \frac{m}{k} \begin{bmatrix} 0.75 & 0.53033006 & 0.30618620 & 0.43301266 \\ 0.53033006 & 1.3749999 & 0.79385662 & 1.1226827 \\ 0.30618620 & 0.79385662 & 1.125 & 1.5909899 \\ 0.43301266 & 1.1226827 & 1.5909899 & 5.2499990 \end{bmatrix}$$

$\lambda = 1/\omega^2$

and $\vec{Y} = [U] \vec{X}$

(7.31) From Eq. (7.84),

$$u_{11} = \sqrt{a_{11}} = \sqrt{16} = 4, \quad u_{12} = \frac{a_{12}}{u_{11}} = -\frac{20}{4} = -5, \quad u_{13} = \frac{a_{13}}{u_{11}} = -\frac{24}{4} = -6$$

$$u_{22} = \left(a_{22} - u_{12}^2\right)^{1/2} = (89 - 25)^{1/2} = 8$$

$$u_{23} = \frac{1}{u_{22}}\left(a_{23} - u_{12}\, u_{13}\right) = \frac{1}{8}\left(-50 - (-5)(-6)\right) = -10$$

$$u_{33} = \left(a_{33} - u_{13}^2 - u_{23}^2\right)^{1/2} = (280 - 36 - 100)^{1/2} = 12$$

$$[A] = [U]^T [U]$$

with $$[U] = \begin{bmatrix} 4 & -5 & -6 \\ 0 & 8 & -10 \\ 0 & 0 & 12 \end{bmatrix}$$

(7.32) From Example 7.3, $\omega = 6.5591 \times 10^{-4} \sqrt{EI}$

Since $E = 2.07 \times 10^{11}\ N/m^2$, $\omega = 298.421 \sqrt{I}$ rad/sec

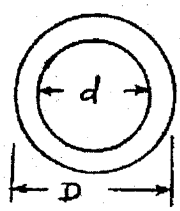
In order to have $\omega = 0.5$ Hz $= 3.1416$ rad/sec,

$$298.421 \sqrt{I} = 3.1416 \quad \Rightarrow \quad I = 0.00011083\ m^4$$

For a tubular section,

$$I = \frac{\pi}{64}\left(D^4 - d^4\right) = 0.00011083$$

i.e. $$D^4 - d^4 = 0.0022578\ m^4$$



To minimize weight, we need to minimize $(D^2 - d^2)$.

Problem is: Find D and d to minimize $(D^2 - d^2)$

subject to $D^4 - d^4 = 0.0022578$

or Find d and $r = D/d$

to minimize $f = d^2 (r^2 - 1)$ (E₁)

subject to $d^4 (r^4 - 1) = 0.0022578$ (E₂)

Eq. (E₂) gives $$d^2 = \frac{0.047516}{\sqrt{r^4 - 1}}$$

Eq. (E₁) becomes $$f = \frac{0.047516\ (r^2 - 1)}{\sqrt{r^4 - 1}}$$

For minimum of f, $$\frac{df}{dr} = \frac{d}{dr}\left(\frac{r^2 - 1}{\sqrt{r^4 - 1}}\right) = 0$$

i.e., $$\frac{(r^2 - 1)\,\frac{1}{2}\,(r^4 - 1)^{-\frac{1}{2}}\,(4r^3) - \sqrt{r^4 - 1}\ (2r)}{r^4 - 1} = 0$$

i.e., $r^2(r^2-1) = (r^2+1)(r^2-1)$

i.e., $r^2=1$ or $r^2 = r^2+1$

i.e., $r=1$ is the only feasible solution.

i.e., A solid circular section.

Since $I = \dfrac{\pi D^4}{64} = 0.00011083$, $D = 0.21798$ m

---

(7.33) The problem-dependent data for Program 13 and the output are given below.

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION D(3,3),E(3,3)
      DATA N,ITMAX,EPS/3,200,1.0E-05/
      DATA D/2.5,-1.0,0.0,-1.0,5.0,-1.41421356,0.0,-1.41421356,10.0/
C END OF PROBLEM-DEPENDENT DATA
```

```
     EIGENVALUE SOLUTION BY JACOBI METHOD

     GIVEN MATRIX
       0.250000E+01  -0.100000E+01   0.000000E+00
      -0.100000E+01   0.500000E+01  -0.141421E+01
       0.000000E+00  -0.141421E+01   0.100000E+02

      EIGEN VALUES ARE
       0.103807E+02   0.500000E+01   0.211932E+01

      EIGEN VECTORS
          FIRST          SECOND          THIRD
        0.329706E-01   0.359223E+00   0.932669E+00
       -0.259784E+00  -0.898022E+00   0.355062E+00
        0.965104E+00  -0.253999E+00   0.637121E-01
```

(7.34) The problem-dependent data for Program 16 and output are given.

```
C FOLLOWING 5 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION BK(3,3),BM(3,3),U(3,3),UI(3,3),UT1(3,3),BMU(3,3),
     2 UMU(3,3),XF(3,3),EV(3,3)
      DATA BK/2.,-1.,0.,-1.,3.,-2.,0.,-2.,2./
      DATA BM/1.,0.,0.,0.,1.,0.,0.,0.,2./
      ND=3
C END OF PROBLEM-DEPENDENT DATA
```

```
UPPER TRIANGULAR MATRIX [U]:

   0.141421E+01  -0.707107E+00   0.000000E+00
   0.000000E+00   0.158114E+01  -0.126491E+01
   0.000000E+00   0.000000E+00   0.632456E+00
```

```
INVERSE OF THE UPPER TRIANGULAR MATRIX, [UI],

   0.707107E+00     0.316228E+00     0.632456E+00
   0.000000E+00     0.632456E+00     0.126491E+01
   0.000000E+00     0.000000E+00     0.158114E+01

MATRIX [UMU] = [UTI][M][UI]:

   0.500000E+00     0.223607E+00     0.447214E+00
   0.223607E+00     0.500000E+00     0.100000E+01
   0.447214E+00     0.100000E+01     0.700000E+01

EIGENVALUES:

   0.718421E+01     0.572771E+00     0.243019E+00

EIGENVECTORS (COLUMNWISE):

   0.721309E+00    -0.664609E+00     0.194954E+00
   0.134220E+01    -0.168876E+00    -0.412289E+00
   0.155923E+01     0.226436E+00     0.132344E+00
```

7.35 The problem-dependent data and results are given (Program 16).

```
C FOLLOWING 5 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION BK(4,4),BM(4,4),U(4,4),UI(4,4),UTI(4,4),BMU(4,4),
    2 UMU(4,4),XF(4,4),EV(4,4)
      DATA BK/4.,-2.,0.,0.,-2.,3.,-1.,0.,0.,-1.,2.,-1.,0.,0.,-1.,1./
      DATA BM/3.,0.,0.,0.,0.,2.,0.,0.,0.,0.,1.,0.,0.,0.,0.,1./
      ND=4
C END OF PROBLEM-DEPENDENT DATA
```

```
UPPER TRIANGULAR MATRIX [U]:

   0.200000E+01    -0.100000E+01     0.000000E+00     0.000000E+00
   0.000000E+00     0.141421E+01    -0.707107E+00     0.000000E+00
   0.000000E+00     0.000000E+00     0.122474E+01    -0.816497E+00
   0.000000E+00     0.000000E+00     0.000000E+00     0.577350E+00

INVERSE OF THE UPPER TRIANGULAR MATRIX, [UI],

   0.500000E+00     0.353553E+00     0.204124E+00     0.288675E+00
   0.000000E+00     0.707107E+00     0.408248E+00     0.577350E+00
   0.000000E+00     0.000000E+00     0.816497E+00     0.115470E+01
   0.000000E+00     0.000000E+00     0.000000E+00     0.173205E+01

MATRIX [UMU] = [UTI][M][UI]:

   0.750000E+00     0.530330E+00     0.306186E+00     0.433013E+00
   0.530330E+00     0.137500E+01     0.793857E+00     0.112268E+01
   0.306186E+00     0.793857E+00     0.112500E+01     0.159099E+01
   0.433013E+00     0.112268E+01     0.159099E+01     0.525000E+01

EIGENVALUES:

   0.623190E+01     0.143190E+01     0.500000E+00     0.336191E+00
```

```
EIGENVECTORS (COLUMNWISE):

     0.480456E+00    -0.437028E+00     0.267261E+00    -0.821001E-01
     0.845263E+00    -0.416241E+00    -0.267258E+00     0.202104E+00
     0.130360E+01     0.206727E+00    -0.267263E+00    -0.431800E+00
     0.155276E+01     0.685329E+00     0.267266E+00     0.218693E+00
```

7.36 The problem-dependent data and output are given (Program 16).

```
C FOLLOWING 5 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION BK(3,3),BM(3,3),U(3,3),UI(3,3),UTI(3,3),BMU(3,3),
    2 UMU(3,3),XP(3,3),EV(3,3)
      DATA BK/2.,-1.,0.,-1.,2.,-1.,0.,-1.,3./
      DATA BM/1.,0.,0.,0.,1.,0.,0.,0.,2./
      ND=3
C END OF PROBLEM-DEPENDENT DATA
  UPPER TRIANGULAR MATRIX [U]:

    0.141421E+01    -0.707107E+00     0.000000E+00
    0.000000E+00     0.122474E+01    -0.816497E+00
    0.000000E+00     0.000000E+00     0.152753E+01

  INVERSE OF THE UPPER TRIANGULAR MATRIX, [UI],

    0.707107E+00     0.408248E+00     0.218218E+00
    0.000000E+00     0.816497E+00     0.436436E+00
    0.000000E+00     0.000000E+00     0.654654E+00

  MATRIX [UMU] = [UTI][M][UI]:

    0.500000E+00     0.288675E+00     0.154303E+00
    0.288675E+00     0.833333E+00     0.445435E+00
    0.154303E+00     0.445435E+00     0.109524E+01

  EIGENVALUES:

    0.151668E+01     0.595606E+00     0.316285E+00

  EIGENVECTORS (COLUMNWISE):

    0.610531E+00    -0.468617E+00     0.349193E+00
    0.818509E+00    -0.150438E+00    -0.405654E+00
    0.486814E+00     0.420340E+00     0.122055E+00
```

7.37 The main program which calls the subroutine MITER and output are given.

```
C **=================================================================
C
C PROGRAM 14
C MAIN PROGRAM FOR CALLING THE SUBROUTINE MITER
C
C **=================================================================
C FOLLOWING 8 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION D(3,3),X(3),XS(3),B(3,3),C(3,3),XX(3),XM(3,3),FREQ(3),
    2 EIG(3,3)
```

```
      N=3
      NVEC=3
      DATA D/3.,2.,1.,2.,2.,1.,1.,1.,1./
      DATA XM/1.,0.,0.,0.,1.,0.,0.,0.,1./
      DATA XS/1.,1.,1./
      EPS=0.00001
C END OF PROBLEM-DEPENDENT DATA
      CALL MITER (D,X,XS,N,NVEC,B,C,XX,XM,EPS,FREQ,EIG)
      WRITE (18,10)
10    FORMAT (//,34H SOLUTION OF EIGENVALUE PROBLEM BY,/,
     2 24H MATRIX ITERATION METHOD)
      WRITE (18,20) (FREQ(I),I=1,NVEC)
20    FORMAT (//,20H NATURAL FREQUENCIES,//,3(E15.8,1X))
      WRITE (18,30)
30    FORMAT (//,26H MODE SHAPES (COLUMNWISE):,/)
      DO 40 I=1,NVEC
40    WRITE (18,50) (EIG(I,J),J=1,N)
50    FORMAT (3(E15.8,1X),/)
      STOP
      END
```

```
ITER=    0   0.10000000E+01 0.10000000E+01 0.10000000E+01
ITER=    1   0.10000000E+01 0.83333331E+00 0.50000000E+00
ITER=    2   0.10000000E+01 0.80645168E+00 0.45161295E+00
ITER=    3   0.10000000E+01 0.80254781E+00 0.44585988E+00
ITER=    4   0.10000000E+01 0.80201757E+00 0.44514498E+00
ITER=    5   0.10000000E+01 0.80194795E+00 0.44505492E+00
ITER=    6   0.10000000E+01 0.80193895E+00 0.44504350E+00
ITER=    7   0.10000000E+01 0.80193788E+00 0.44504207E+00

FREQ= 0.44504169E+00

ITER=    0   0.10000000E+01 0.10000000E+01 0.10000000E+01
ITER=    1   0.10000000E+01-0.36259139E+00-0.15934708E+01
ITER=    2   0.10000000E+01-0.46800232E+00-0.14036615E+01
ITER=    3   0.10000000E+01-0.51440042E+00-0.13200551E+01
ITER=    4   0.10000000E+01-0.53577375E+00-0.12815419E+01
ITER=    5   0.10000000E+01-0.54582363E+00-0.12634329E+01
ITER=    6   0.10000000E+01-0.55059475E+00-0.12548355E+01
ITER=    7   0.10000000E+01-0.55287021E+00-0.12507354E+01
ITER=    8   0.10000000E+01-0.55395770E+00-0.12487757E+01
ITER=    9   0.10000000E+01-0.55447805E+00-0.12478381E+01
ITER=   10   0.10000000E+01-0.55472714E+00-0.12473894E+01
ITER=   11   0.10000000E+01-0.55484635E+00-0.12471745E+01
ITER=   12   0.10000000E+01-0.55490345E+00-0.12470715E+01
ITER=   13   0.10000000E+01-0.55493081E+00-0.12470224E+01
ITER=   14   0.10000000E+01-0.55494392E+00-0.12469988E+01
ITER=   15   0.10000000E+01-0.55495018E+00-0.12469875E+01
ITER=   16   0.10000000E+01-0.55495316E+00-0.12469820E+01

FREQ= 0.12469784E+01

ITER=    0   0.10000000E+01 0.10000000E+01 0.10000000E+01
ITER=    1   0.10000000E+01-0.22478061E+01 0.18022429E+01
ITER=    2   0.10000000E+01-0.22469645E+01 0.18019053E+01
ITER=    3   0.10000000E+01-0.22469645E+01 0.18019053E+01
```

```
FREQ= 0.18019379E+01


SOLUTION OF EIGENVALUE PROBLEM BY
MATRIX ITERATION METHOD


NATURAL FREQUENCIES

0.44504169E+00  0.12469784E+01  0.18019379E+01


MODE SHAPES (COLUMNWISE):

0.10000000E+01  0.10000000E+01  0.10000000E+01

0.80193788E+00 -0.55495316E+00 -0.22469645E+01

0.44504207E+00 -0.12469820E+01  0.18019053E+01
```

(7.38) *The problem-dependent data for Program 13 and results are given.*

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION D(3,3),E(3,3)
      DATA N,ITMAX,EPS/3,200,1.0E-05/
      DATA D/5.,-1.,1.,-1.,6.,-4.,1.,-4.,3./
C END OF PROBLEM-DEPENDENT DATA
EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX
  0.500000E+01  -0.100000E+01   0.100000E+01
 -0.100000E+01   0.600000E+01  -0.400000E+01
  0.100000E+01  -0.400000E+01   0.300000E+01

EIGEN VALUES ARE
  0.923048E+01   0.455549E+01   0.214034E+00

EIGEN VECTORS
    FIRST          SECOND          THIRD
  0.312915E+00   0.948172E+00  -0.552614E-01
 -0.775620E+00   0.288684E+00   0.561315E+00
  0.548176E+00  -0.132782E+00   0.825755E+00
```

(7.39) *The problem-dependent data for Program 13 and results are given.*

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION D(3,3),E(3,3)
      DATA N,ITMAX,EPS/3,200,1.0E-05/
      DATA D/16.,-20.,-24.,-20.,89.,-50.,-24.,-50.,280./
C END OF PROBLEM-DEPENDENT DATA
```

EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX

0.160000E+02	-0.200000E+02	-0.240000E+02
-0.200000E+02	0.890000E+02	-0.500000E+02
-0.240000E+02	-0.500000E+02	0.280000E+03

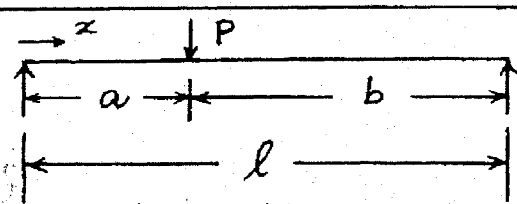
EIGEN VALUES ARE

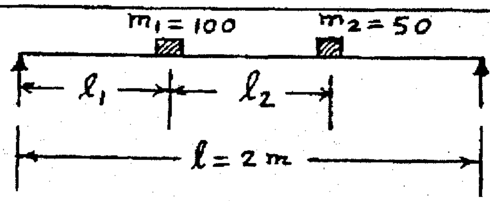
0.293547E+03	0.855832E+02	0.586943E+01

EIGEN VECTORS

FIRST	SECOND	THIRD
0.673506E-01	-0.332765E+00	0.940601E+00
~~0.230709E+00~~	~~0.922386E+00~~	~~0.309802E+00~~
-0.970689E+00	0.196140E+00	0.138895E+00

7.40





Basic relationship:

$$w(x) = \begin{cases} \dfrac{P b x}{6EI\ell}\left(\ell^2 - b^2 - x^2\right); & 0 \le x \le a \qquad (E_1) \\[2ex] -\dfrac{P a(\ell - x)}{6EI\ell}\left(a^2 + x^2 - 2\ell x\right); & a \le x \le \ell \qquad (E_2) \end{cases}$$

Deflection of mass m_1 due to load $m_1 g$:

Using $x = \ell_1$, $b = 2 - \ell_1$ and $\ell = 2$ in (E_1):

$$w_1' = \frac{(100 \times 9.81)(2 - \ell_1)\ell_1}{6EI(2)}\left\{4 - (2 - \ell_1)^2 - \ell_1^2\right\} = \frac{981\, \ell_1^2 (2 - \ell_1)^2}{6EI} \qquad (E_3)$$

Deflection of mass m_2 due to load $m_1 g$:

Using $x = \ell_1 + \ell_2$, $a = \ell_1$, $b = 2 - \ell_1$, $\ell = 2$ in (E_2):

$$w_2' = -\frac{(100 \times 9.81)\, \ell_1 (2 - \ell_1 - \ell_2)}{6EI(2)}\left\{\ell_1^2 + (\ell_1 + \ell_2)^2 - 2(2)(\ell_1 + \ell_2)\right\}$$

$$= -\frac{981\, \ell_1 (2 - \ell_1 - \ell_2)}{12EI}\left(2\ell_1^2 + \ell_2^2 + 2\ell_1\ell_2 - 4\ell_1 - 4\ell_2\right) \qquad (E_4)$$

Deflection of mass m_1 due to load $m_2 g$:

Using $x = \ell_1$, $\ell = 2$, $b = (2 - \ell_1 - \ell_2)$ in (E_1):

$$w_1'' = \frac{(50 \times 9.81)(2 - \ell_1 - \ell_2)\, \ell_1}{6EI(2)}\left\{4 - (2 - \ell_1 - \ell_2)^2 - \ell_1^2\right\}$$

$$= \frac{490.5\, \ell_1 (2 - \ell_1 - \ell_2)\left(-2\ell_1^2 - \ell_2^2 + 4\ell_1 + 4\ell_2 - 2\ell_1\ell_2\right)}{12EI} \qquad (E_5)$$

Deflection of mass m_2 due to load $m_2 g$:

Using $x = l_1 + l_2$, $l = 2$ and $b = 2 - l_1 - l_2$ in (E_1):

$$w_2'' = \frac{(50 \times 9.81)(2-l_1-l_2)(l_1+l_2)}{6EI(2)} \left\{ 4 - (2-l_1-l_2)^2 - (l_1+l_2)^2 \right\}$$

$$= \frac{490.5\ (l_1+l_2)(2-l_1-l_2)(-2l_1^2 - 2l_2^2 + 4l_1 + 4l_2 - 4l_1 l_2)}{12\,EI} \qquad (E_6)$$

Total deflection of masses m_1 and m_2 are:

$$w_1 = w_1' + w_1'' = \frac{981\ l_1^2 (2-l_1)^2}{6EI} + \frac{490.5\, l_1\ (2-l_1-l_2)(-2l_1^2 - l_2^2 + 4l_1 + 4l_2 - 2l_1 l_2)}{12\,EI} \qquad (E_7)$$

$$w_2 = w_2' + w_2'' = \frac{-981\ l_1 (2-l_1-l_2)(2l_1^2 + l_2^2 + 2l_1 l_2 - 4l_1 - 4l_2)}{12\,EI} + \frac{490.5\ (l_1+l_2)(2-l_1-l_2)(-2l_1^2 - 2l_2^2 + 4l_1 + 4l_2 - 4l_1 l_2)}{12\,EI} \qquad (E_8)$$

Fundamental natural frequency is given by

$$\omega = \left\{ \frac{g\,(m_1 w_1 + m_2 w_2)}{(m_1 w_1^2 + m_2 w_2^2)} \right\}^{\frac{1}{2}} = 3.1321 \left(\frac{2w_1 + w_2}{2w_1^2 + w_2^2} \right) \qquad (E_9)$$

To maximize ω, we can maximize ω^2.

Problem is: Find l_1 and l_2

to maximize $f = \left(\dfrac{2w_1 + w_2}{2w_1^2 + w_2^2} \right)$

where w_1 and w_2 are given by (E_7) and (E_8).

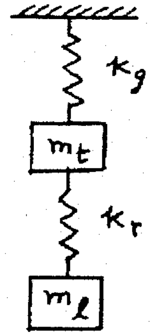
Problem can be solved as follows:

Treat f as a function of l_1 and l_2.

Set $\dfrac{\partial f}{\partial l_1} = 0$ and $\dfrac{\partial f}{\partial l_2} = 0 \qquad (E_{10})$

Solve Eqs. (E_{10}) for l_1 and l_2.

(7.41) Stiffness of one girder (simply supported beam)



$$k_{g_1} = \frac{48\,EI}{\ell^3} = \frac{48\,(30\times10^6)\left(\frac{1}{12}a^4\right)}{(30\times12)^3} = 2.572\ a^4$$

where a = width and depth of cross-section (inch) of girder.

$$k_g = 2\,k_{g_1} = 5.144\ a^4 \quad lb/in$$

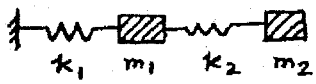
$$m_t = \text{mass of trolley} = \frac{40000}{386.4} = 103.5197\ lb\text{-}sec^2/in$$

$$\text{stiffness of rope} = k_r = \frac{AE}{\ell} = \frac{\frac{\pi d^2}{4}(30\times10^6)}{(20\times12)} = 98175\ d^2$$

where d = diameter of the rope (inch).

$$m_\ell = \text{mass of lifted load} = \frac{10000}{386.4} = 25.8799\ lb\text{-}sec^2/in$$

From section 5.3, the natural frequencies of a 2 d.o.f. system (shown in adjacent figure) are given by



$$\omega_{1,2}^2 = \frac{1}{2}\left\{\frac{(k_1+k_2)\,m_2 + k_2 m_1}{m_1 m_2}\right\} \mp \left[\left\{\frac{(k_1+k_2)\,m_2 + k_2 m_1}{m_1 m_2}\right\}^2 - 4\left\{\frac{(k_1+k_2)\,k_2 - k_2^2}{m_1 m_2}\right\}\right]^{1/2} \quad (E_1)$$

Here $k_1 = k_g$, $k_2 = k_r$, $m_1 = m_t$, $m_2 = m_\ell$ and hence ω_1^2 and ω_2^2 can be expressed as functions of a and d:

$$\omega_1^2 = \omega_1^2(a,d) \quad ; \quad \omega_2^2 = \omega_2^2(a,d) \qquad (E_2)$$

Requirement is

$$\omega_1^2(a,d) > (157.08)^2 \quad ; \quad \omega_2^2(a,d) > (157.08)^2 \qquad (E_3)$$

since 1500 rpm = 157.08 rad/sec.

choose a and d such that the inequalities (E_3) are satisfied. A trial and error procedure can be used for this purpose.

Chapter 8

Continuous Systems

8.1 $c = \left(\frac{P}{\rho}\right)^{1/2} = \left(\frac{4000}{5}\right)^{1/2} = 28.2843 \text{ m/s}$

8.2 ρ = mass density x area = $7830 \left\{\frac{\pi}{4}(2\times10^{-3})^2\right\} = 24.5987\times10^{-3}$ kg/m

$P = 250$ N, $l = 2$ m

(b) $c = (P/\rho)^{1/2} = \left(\frac{250}{24.5987\times10^{-3}}\right)^{1/2} = 100.8124$ m/s

(a) $\omega_1 = \frac{c\pi}{l} = \frac{100.8124\ \pi}{2} = 158.3561$ rad/s $= 25.2031$ Hz

8.3 $\omega_1 = 3000\,(2\pi)$ rad/sec $= \frac{\pi c}{l} = \frac{\pi c}{2}$

$c = (6000\pi \times 2/\pi) = 12000$ m/s

$\omega_3 = 3\pi c/l = 3\,\omega_1 = 9000$ Hz

$c_{original} = (P/\rho)^{1/2} = 12000$ m/s

$c_{new} = (1.2\,P/\rho)^{1/2} = 1.0954\,(P/\rho)^{1/2} = 1.0954\ c_{original}$

$\therefore$ ω_1 and ω_3 are increased by 9.54%

8.4 $P = 30000$ N, $\rho = 2$ kg/m

$c = \left(\frac{P}{\rho}\right)^{1/2} = (30000/2)^{1/2} = 122.4745$ m/s

Time taken $= \frac{300}{122.4745} = 2.4495$ s

8.5 At $x = 0$: $$P\,\frac{\partial w}{\partial x} = m\,\frac{\partial^2 w}{\partial t^2} \qquad (E_1)$$

At $x = l$: $$P\,\frac{\partial w}{\partial x} = -k\,w \qquad (E_2)$$

General solution is

$$w(x,t) = W(x)\cdot T(t) = \left(A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right) \qquad (E_3)$$

Equations (E_1) and (E_3) give:

$$\left\{P\left(-A\,\frac{\omega}{c}\sin\frac{\omega x}{c} + B\,\frac{\omega}{c}\cos\frac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right) = -m\,\omega^2\left(A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right)\right\}_{x=0}$$

i.e., $$A\,(m\omega^2) + B\left(\frac{P\omega}{c}\right) = 0 \qquad (E_4)$$

Chapter 8

Continuous Systems

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$c = (6000\pi \times 2/\pi) = 12000$ m/s

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General solution is

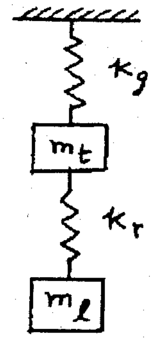
$$w(x,t) = W(x)\cdot T(t) = \left(A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right) \qquad (E_3)$$

Equations (E_1) and (E_3) give:

$$\Big\{P\left(-A\frac{\omega}{c}\sin\frac{\omega x}{c} + B\frac{\omega}{c}\cos\frac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right)$$
$$= -m\omega^2\left(A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right)\Big\}_{x=0}$$

i.e., $\quad A\,(m\omega^2) + B\left(\frac{P\omega}{c}\right) = 0 \qquad (E_4)$

(7.41) Stiffness of one girder (simply supported beam)



$$k_{g_1} = \frac{48EI}{\ell^3} = \frac{48(30\times10^6)(\frac{1}{12}a^4)}{(30\times12)^3} = 2.572\ a^4$$

where a = width and depth of cross-section (inch) of girder.

$$k_g = 2\,k_{g_1} = 5.144\ a^4 \quad \text{lb/in}$$

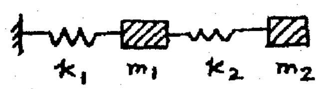
m_t = mass of trolley = $\frac{40000}{386.4}$ = 103.5197 lb-sec²/in

stiffness of rope = $k_r = \frac{AE}{\ell} = \frac{\frac{\pi d^2}{4}(30\times10^6)}{(20\times12)} = 98175\ d^2$

where d = diameter of the rope (inch).

m_ℓ = mass of lifted load = $\frac{10000}{386.4}$ = 25.8799 lb-sec²/in

From section 5.3, the natural frequencies of a 2 d.o.f. system (shown in adjacent figure) are given by



$$\omega_{1,2}^2 = \frac{1}{2}\left\{\frac{(k_1+k_2)m_2 + k_2 m_1}{m_1 m_2}\right\} \mp \left[\left\{\frac{(k_1+k_2)m_2 + k_2 m_1}{m_1 m_2}\right\}^2 - 4\left\{\frac{(k_1+k_2)k_2 - k_2^2}{m_1 m_2}\right\}\right]^{1/2} \qquad (E_1)$$

Here $k_1 = k_g$, $k_2 = k_r$, $m_1 = m_t$, $m_2 = m_\ell$ and hence ω_1^2 and ω_2^2 can be expressed as functions of a and d:

$$\omega_1^2 = \omega_1^2(a, d) \quad ; \quad \omega_2^2 = \omega_2^2(a, d) \qquad (E_2)$$

Requirement is

$$\omega_1^2(a, d) > (157.08)^2 \quad ; \quad \omega_2^2(a, d) > (157.08)^2 \qquad (E_3)$$

since 1500 rpm = 157.08 rad/sec.

Choose a and d such that the inequalities (E_3) are satisfied. A trial and error procedure can be used for this purpose.

Eqs. (E_2) and (E_3) yield

$$P\left(-\frac{\omega}{c} A \sin\frac{\omega l}{c} + B\frac{\omega}{c}\cos\frac{\omega l}{c}\right) = -k\left(A\cos\frac{\omega l}{c} + B\sin\frac{\omega l}{c}\right)$$

i.e.,

$$A\left(-\frac{P\omega}{c}\sin\frac{\omega l}{c} + k\cos\frac{\omega l}{c}\right) + B\left(\frac{P\omega}{c}\cos\frac{\omega l}{c} + k\sin\frac{\omega l}{c}\right) = 0 \qquad (E_5)$$

Eqs. (E_4) and (E_5) give the frequency equation:

$$\begin{vmatrix} (m\omega^2) & (P\omega/c) \\ \left(-\frac{P\omega}{c}\sin\frac{\omega l}{c} + k\cos\frac{\omega l}{c}\right) & \left(\frac{P\omega}{c}\cos\frac{\omega l}{c} + k\sin\frac{\omega l}{c}\right) \end{vmatrix} = 0$$

which, upon simplification, becomes

$$\tan\alpha = \left\{\frac{P k - \left(\frac{P m c^2}{l^2}\right)\alpha^2}{\left(\frac{c^2 m k}{l}\right)\alpha + \left(\frac{P^2 c}{l}\right)\alpha}\right\} \qquad (E_6)$$

8.6

$l = 2$ m, $d = 0.5$ mm, $\rho = 7800$ kg/m^3, $\omega_n = \frac{n c \pi}{l}$, $c = \sqrt{\frac{P}{\rho}}$

(a) $\omega_1 = 1(2\pi)$ rad/sec $= \frac{\pi c}{l} = \frac{\pi}{2}\sqrt{\frac{P}{7800}}$

i.e., $(2\pi)^2 = \frac{\pi^2}{4}\left(\frac{P}{7800}\right)$

i.e., $P = 124{,}800$ N

(b) $\omega_1 = 5(2\pi)$ rad/sec $= \frac{\pi c}{l} = \frac{\pi}{2}\sqrt{\frac{P}{7800}}$

i.e., $100\pi^2 = \frac{\pi^2}{4(7800)} P$

i.e., $P = 3{,}120{,}000$ N

8.7

Let $x = 0$ be fixed and $x = l$ be connected to the pin which can move in a frictionless slot.

$$W(x) = A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}$$

$w(0,t) = 0 \Rightarrow W(0) = 0 \Rightarrow A = 0$

$\frac{\partial w}{\partial x}(l,t) = 0 \Rightarrow \frac{dW}{dx}(l) = 0 \Rightarrow B\cos\frac{\omega l}{c} = 0 \Rightarrow \cos\frac{\omega l}{c} = 0$

$\therefore \quad \frac{\omega l}{c} = (2n+1)\frac{\pi}{2} \quad ; \; n = 0, 1, 2, \ldots$

or $\quad \omega_n = \frac{(2n+1)\pi c}{2l} \quad ; \; n = 0, 1, 2, \ldots$

8.8

$$w(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left[C_n \cos\frac{n c\pi t}{l} + D_n \sin\frac{n c \pi t}{l}\right]$$

where

$$C_n = \frac{2}{l}\int_0^l w_0(x)\sin\frac{n\pi x}{l}\,dx, \qquad D_n = \frac{2}{\pi c n}\int_0^l \dot{w}_0(x)\sin\frac{n\pi x}{l}\,dx$$

Since $w_0(x) = w(x,0) = 0, \quad C_n = 0$

$$D_n = \frac{2}{\pi c n}\left[\int_0^{l/2} \frac{2ax}{l} \sin\frac{n\pi x}{l}\, dx + \int_{l/2}^{l} 2a\left(1-\frac{x}{l}\right) \sin\frac{n\pi x}{l}\, dx\right]$$

$$= \frac{8al}{\pi^3 c n^3} \sin\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} \dfrac{8al}{\pi^3 n^3 c} & \text{if } n \text{ is odd} \end{cases}$$

$$w(x,t) = \frac{8al}{\pi^3 c} \sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^3} \sin\frac{n\pi x}{l} \sin\frac{n\pi c t}{l}$$

8.9 Eq. (8.18) is $\quad c^2 \dfrac{d^2 W}{dx^2} = a\, W(x)$

Multiply this equation by $W(x)$ and integrate from 0 to l:

$$c^2 \int_0^l W(x) \frac{d^2 W(x)}{dx^2}\, dx = a \int_0^l [W(x)]^2\, dx$$

This shows that the sign of a will be same as that of the integral on the left side. Integration by parts gives

$$I = \int_0^l W(x) \cdot \frac{d^2 W(x)}{dx^2}\, dx = W(l) \cdot \frac{dW}{dx}(l) - W(0) \frac{dW}{dx}(0) - \int_0^l \left[\frac{dW}{dx}(x)\right]^2 dx$$

Since the first two terms on the right side of this equation are zero for common boundary conditions, I and hence a will be negative.

Common boundary conditions (examples):

Fixed at ends: $\quad W(0) = W(l) = 0$

Free at ends: $\quad \dfrac{dW}{dx}(0) = \dfrac{dW}{dx}(l) = 0$

One end fixed and other end free: $\quad W(0) = \dfrac{dW}{dx}(l) = 0$

One end fixed and other end connected to a spring:

$$W(0) = 0; \quad \frac{dW}{dx}(l) = -k\, W(l)$$

8.10 $l = 2000$ m, $\quad \rho = 8890$ kg/m^3, $\quad 0 \le \omega_1$ to $\omega_4 \le 20$ Hz

$$\omega_n = \frac{n c \pi}{l} = \frac{n\pi}{l}\sqrt{\frac{P}{\rho}}$$

For $\omega_4 \le 20(2\pi)$ rad/sec, $\quad \dfrac{4\pi}{l}\sqrt{\dfrac{P}{\rho}} \le 40\pi$

i.e., $\quad \sqrt{\dfrac{P}{8890}} \le \dfrac{40\pi\,(2000)}{4\pi}$

i.e., $\quad P \le 35\,560 \times 10^8$ N

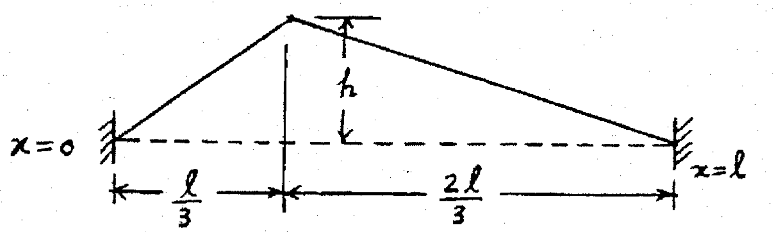
Let the permissible (yield) stress of the material be 300 MPa $= 3 \times 10^8$ N/m^2.

Assuming the diameter of cable as 0.1 m, area of cross-section is $\frac{\pi}{4}(0.1)^2 = 0.007854\ m^2$, and the permissible tension (with factor of safety of one) is

$$(0.007854)(3 \times 10^8) = 2,356,200\ N$$

$\therefore$ Initial tension $= 2.3562 \times 10^6$ N

8.11



Solution is given by Eq. (8.30):

$$w(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \cos \frac{nc\pi t}{l} \qquad (E_1)$$

where

$$C_n = \frac{2}{l} \int_0^l w_0(x) \sin \frac{n\pi x}{l}\, dx \qquad (E_2)$$

The initial deflection $w_0(x)$ is given by

$$w_0(x) = \begin{cases} 3hx/l & \text{for } 0 \le x \le l/3 \\ 3h(l-x)/2l & \text{for } l/3 \le x \le l \end{cases} \qquad (E_3)$$

Substitution of Eq. (E_3) into (E_2) gives

$$C_n = \frac{2}{l}\left\{\int_0^{l/3} \frac{3hx}{l} \sin \frac{n\pi x}{l}\, dx + \int_{l/3}^{l} \frac{3h(l-x)}{2l} \sin \frac{n\pi x}{l}\, dx\right\}$$

$$= \frac{6h}{l^2}\left\{\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} - \frac{xl}{n\pi} \cos \frac{n\pi x}{l}\right\}_0^{l/3} - \frac{3h}{l}\left(\frac{l}{n\pi}\right)\left(\cos \frac{n\pi x}{l}\right)_{l/3}^{l}$$

$$- \frac{3h}{l^2}\left\{\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} - \frac{xl}{n\pi} \cos \frac{n\pi x}{l}\right\}_{l/3}^{l}$$

$$= \frac{9h}{n^2\pi^2} \sin \frac{n\pi}{3}$$

$$= \frac{9h\sqrt{3}}{2\pi^2} \text{ for } n=1, \quad \frac{9h\sqrt{3}}{8\pi^2} \text{ for } n=2, \quad 0 \text{ for } n=3,$$

$$- \frac{9h\sqrt{3}}{32\pi^2} \text{ for } n=4, \quad - \frac{9h\sqrt{3}}{50\pi^2} \text{ for } n=5, \quad 0 \text{ for } n=6, \ldots$$

$$w(x,t) = \frac{9\sqrt{3}}{2\pi^2} h \sin \frac{\pi x}{l} \cos \frac{c\pi t}{l} + \frac{9\sqrt{3}}{8\pi^2} h \sin \frac{2\pi x}{l} \cos \frac{2c\pi t}{l}$$

$$- \frac{9\sqrt{3}}{32\pi^2} h \sin \frac{4\pi x}{l} \cos \frac{4c\pi t}{l} - \ldots$$

At $t=0$:

$$w(x,0) = \left\{ \frac{9\sqrt{3}}{2\pi^2} h \sin\frac{\pi x}{\ell} + \frac{9\sqrt{3}}{8\pi^2} h \sin\frac{2\pi x}{\ell} - \frac{9\sqrt{3}}{32\pi^2} h \sin\frac{4\pi x}{\ell} - \frac{9\sqrt{3}}{50\pi^2} h \sin\frac{5\pi x}{\ell} \right\}$$

At $t = \frac{\ell}{4c}$:

$$w\left(x, \frac{\ell}{4c}\right) = \frac{9\sqrt{3}}{2\pi^2}\frac{h}{\sqrt{2}} \sin\frac{\pi x}{\ell} - \frac{9\sqrt{3}}{32\pi^2} h \sin\frac{4\pi x}{\ell} + \frac{9\sqrt{3}}{50\pi^2}\frac{h}{\sqrt{2}} \sin\frac{5\pi x}{\ell}$$

At $t = \frac{\ell}{3c}$:

$$w\left(x, \frac{\ell}{3c}\right) = \frac{9\sqrt{3}\,h}{4\pi^2}\left\{ \sin\frac{\pi x}{\ell} - \frac{1}{4}\sin\frac{2\pi x}{\ell} + \frac{1}{16}\sin\frac{4\pi x}{\ell} - \frac{1}{25}\sin\frac{5\pi x}{\ell} \right\}$$

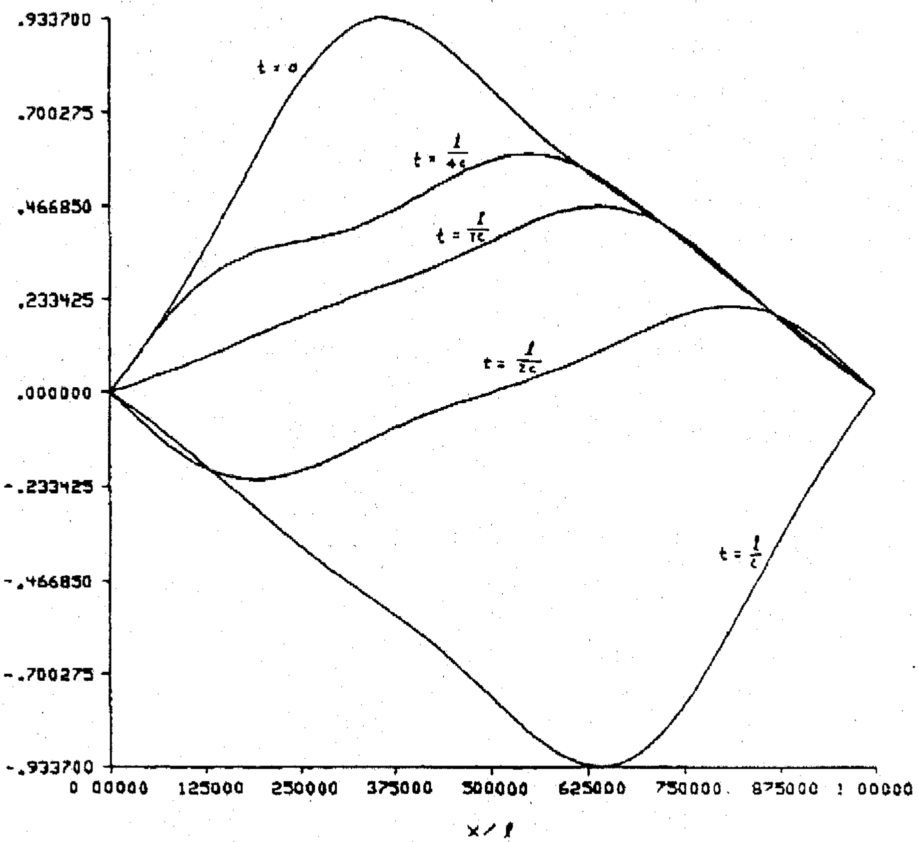
At $t = \frac{\ell}{2c}$:

$$w\left(x, \frac{\ell}{2c}\right) = -\frac{9\sqrt{3}\,h}{8\pi^2}\left\{ \sin\frac{2\pi x}{\ell} + \frac{1}{4}\sin\frac{4\pi x}{\ell} \right\}$$

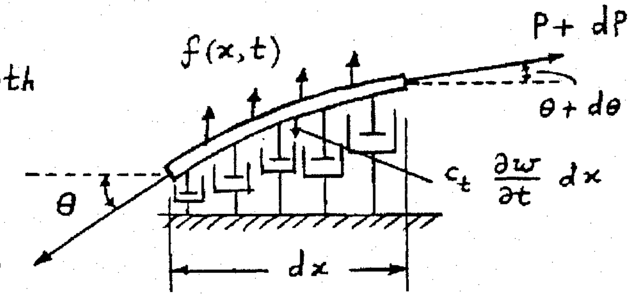
At $t = \frac{\ell}{c}$:

$$w\left(x, \frac{\ell}{c}\right) = \frac{9\sqrt{3}\,h}{2\pi^2}\left\{ -\sin\frac{\pi x}{\ell} + \frac{1}{4}\sin\frac{2\pi x}{\ell} - \frac{1}{16}\sin\frac{4\pi x}{\ell} + \frac{1}{25}\sin\frac{5\pi x}{\ell} \right\}$$

These deflection shapes are shown below:



8.12 c_t = viscous damping coefficient
= force / unit velocity / unit length



$$(P + dP)\sin(\theta + d\theta) + f(x,t)\,dx - c_t \frac{\partial w}{\partial t} dx - P \sin\theta = \rho\, dx\, \frac{\partial^2 w}{\partial t^2} \quad \ldots (E_1)$$

with $dP = \frac{\partial P}{\partial x} dx$ and $\sin(\theta + d\theta) \approx \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx$

Eq. (E_1) can be rewritten as

$$\frac{\partial}{\partial x}\left(P \frac{\partial w}{\partial x}\right) + f(x,t) = \rho(x) \frac{\partial^2 w(x,t)}{\partial t^2} + c_t \frac{\partial w(x,t)}{\partial t}$$

8.13 Free vibration solution is given by

$$w(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left\{C_n \cos\frac{n c \pi t}{l} + D_n \sin\frac{n c \pi t}{l}\right\}$$

with $C_n = \frac{2}{l}\int_0^l w_0(x) \sin\frac{n\pi x}{l}\, dx$

and $D_n = \frac{2}{n c \pi}\int_0^l \dot{w}_0(x) \sin\frac{n\pi x}{l}\, dx$

Since $\dot{w}_0(x) = \frac{\partial w}{\partial t}(x, t=0) = 0$, $D_n = 0$.

For $w_0(x) = w_0 \sin\frac{\pi x}{l}$, $C_n = \frac{2 w_0}{l}\int_0^l \sin\frac{\pi x}{l} \sin\frac{n\pi x}{l}\, dx$

Using the relations

$$\int \sin m_1 x \sin n_1 x\, dx = \frac{\sin(m_1 - n_1)x}{2(m_1 - n_1)} - \frac{\sin(m_1 + n_1)x}{2(m_1 + n_1)}; \quad m_1 \neq n_1$$

$$\int \sin^2 m_1 x\, dx = \frac{x}{2} - \frac{1}{4 m_1}\sin 2 m_1 x,$$

we get for $n = 1$,

$$C_1 = \frac{2 w_0}{l}\int_0^l \sin^2\frac{\pi x}{l}\, dx = w_0$$

and for $n = 2, 3, \ldots,$

$$C_n = \frac{2 w_0}{l}\left\{\frac{\sin\frac{\pi}{l}(n-1)x}{2\frac{\pi}{l}(n-1)} - \frac{\sin\frac{\pi}{l}(n+1)x}{2\frac{\pi}{l}(n+1)}\right\}_0^l = 0$$

$$\therefore \quad w(x,t) = w_0 \sin\frac{\pi x}{l} \cos\frac{c\pi t}{l}$$

8.14

$$u(x,t) = \left(A \cos\frac{\omega x}{c} + B \sin\frac{\omega x}{c}\right)\left(C \cos\omega t + D \sin\omega t\right)$$

$$\frac{\partial u}{\partial x} = \frac{\omega}{c}\left(-A \sin\frac{\omega x}{c} + B \cos\frac{\omega x}{c}\right)\left(C\cos\omega t + D \sin\omega t\right)$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \;\Rightarrow\; B = 0$$

$$\frac{\partial u}{\partial x}(l,t) = 0 \;\Rightarrow\; -\frac{\omega}{c} A \sin\frac{\omega l}{c}\left(C\cos\omega t + D\sin\omega t\right) = 0$$

$$\Rightarrow\; \sin\frac{\omega l}{c} = 0 \;;\qquad \frac{\omega_n l}{c} = n\pi$$

$$\omega_n = \frac{n\pi c}{l} = \frac{n\pi}{l}\sqrt{\frac{E}{\rho}}$$

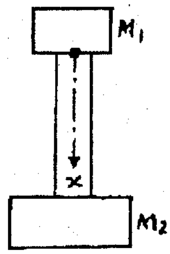
$$u(x,t) = \sum_{n=1}^{\infty} \cos\frac{n\pi x}{l}\left[C_n \cos\frac{n\pi c t}{l} + D_n \sin\frac{n\pi c t}{l}\right]$$

where $C_n = \dfrac{2}{l}\displaystyle\int_0^l u_0(x)\cos\frac{n\pi x}{l}\,dx$

and $D_n = \dfrac{2}{n\pi c}\displaystyle\int_0^l \dot{u}_0(x)\cos\frac{n\pi x}{l}\,dx$

8.15

(a) $u(x,t) = \left(A \cos\dfrac{\omega x}{c} + B \sin\dfrac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right)$



At $x=0$: $\quad M_1 \dfrac{\partial^2 u}{\partial t^2} = AE\dfrac{\partial u}{\partial x}$

$$-\omega^2 M_1 A = AE\frac{\omega}{c} B \;\Rightarrow\; B = -\left(\frac{\omega M_1 c}{AE}\right) A$$

At $x = l$: $\quad M_2 \dfrac{\partial^2 u}{\partial t^2} = -AE\dfrac{\partial u}{\partial x}$

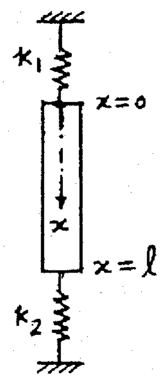
$$-\omega^2 M_2\left(A\cos\frac{\omega l}{c} + B\sin\frac{\omega l}{c}\right) = -AE\frac{\omega}{c}\left(-A\sin\frac{\omega l}{c} + B\cos\frac{\omega l}{c}\right)$$

i.e. $$A\left(-\omega^2 M_2\cos\frac{\omega l}{c} - AE\frac{\omega}{c}\sin\frac{\omega l}{c}\right) = B\left(-\frac{AE\omega}{c}\cos\frac{\omega l}{c} + \omega^2 M_2\sin\frac{\omega l}{c}\right)$$

i.e. $$\omega^2 M_2\cos\frac{\omega l}{c} + \frac{AE\omega}{c}\sin\frac{\omega l}{c} + \frac{\omega M_1 c}{AE}\left(-\omega^2 M_2\sin\frac{\omega l}{c} + \frac{AE\omega}{c}\cos\frac{\omega l}{c}\right) = 0$$

This is the frequency equation.

(b) $u(x,t) = \left(A\cos\dfrac{\omega x}{c} + B\sin\dfrac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right)$ --- (E_1)



At $x=0$, $\quad k_1 u = AE\dfrac{\partial u}{\partial x} \;\Rightarrow\; B = \dfrac{k_1 c}{AE\omega} A$ --- (E_2)

At $x = l$, $\quad k_2 u = -AE\dfrac{\partial u}{\partial x}$

$$\Rightarrow k_2\left(A\cos\frac{\omega l}{c} + B\sin\frac{\omega l}{c}\right) = -AE\frac{\omega}{c}\left\{-A\sin\frac{\omega l}{c} + B\cos\frac{\omega l}{c}\right\}$$ --- (E_3)

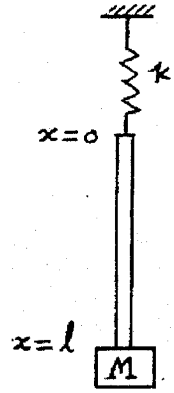
Substituting (E_2) into (E_3), we get

$$A\left[\left(k_2 - \frac{k_1 c}{AE\omega}\right)\cos\frac{\omega l}{c} + \left(\frac{k_1 k_2 c}{AE\omega} - \frac{AE\omega}{c}\right)\sin\frac{\omega l}{c}\right] = 0$$

Hence the frequency equation is given by

$$\left(k_2 - \frac{k_1 c}{AE\omega}\right)\cos\frac{\omega l}{c} + \left(\frac{k_1 k_2 c}{AE\omega} - \frac{AE\omega}{c}\right)\sin\frac{\omega l}{c} = 0$$

i.e., $$\tan\frac{\omega l}{c} = \left(\frac{k_1 c^2 - k_2 AE\omega c}{k_1 k_2 c^2 - A^2E^2\omega^2}\right)$$



(c) $$u(x,t) = \left(\underset{\sim}{A}\cos\frac{\omega x}{c} + \underset{\sim}{B}\sin\frac{\omega x}{c}\right)(C\cos\omega t + D\sin\omega t) \quad \text{--- } (E_1)$$

At $x=0$, $\quad k\,u = AE\dfrac{\partial u}{\partial x} \Rightarrow \underset{\sim}{B} = \dfrac{k c}{AE\omega}\underset{\sim}{A}$ --- (E_2)

At $x=l$, $\quad AE\dfrac{\partial u}{\partial x} = -M\dfrac{\partial^2 u}{\partial t^2}$

$$\Rightarrow AE\left(-\underset{\sim}{A}\frac{\omega}{c}\sin\frac{\omega l}{c} + \underset{\sim}{B}\frac{\omega}{c}\cos\frac{\omega l}{c}\right)$$
$$= M\left(\underset{\sim}{A}\cos\frac{\omega l}{c} + \underset{\sim}{B}\sin\frac{\omega l}{c}\right)\omega^2 \quad \text{--- } (E_3)$$

Substituting (E_2) into (E_3), we get

$$\frac{AE\omega}{c}\underset{\sim}{A}\left(-\sin\frac{\omega l}{c} + \frac{kc}{AE\omega}\cos\frac{\omega l}{c}\right) = M\omega^2\underset{\sim}{A}\left(\cos\frac{\omega l}{c} + \frac{kc}{AE\omega}\sin\frac{\omega l}{c}\right)$$

This gives the frequency equation

$$\tan\frac{\omega l}{c} = \left\{\frac{AE\omega c\,(k - M\omega^2)}{A^2E^2\omega^2 - M\omega^2 k c^2}\right\}$$

(8.16) For a clamped-free bar in longitudinal vibration, fundamental frequency $= \dfrac{\pi c}{2l}$

If the bar is fixed at $x=0$ and attached to a mass M at $x=l$, its fundamental frequency $= \dfrac{\alpha_1 c}{l}$.

Here $\dfrac{\alpha_1 c}{l} = \dfrac{1}{2}\left(\dfrac{\pi c}{2l}\right) \Rightarrow \alpha_1 = \dfrac{\pi}{4} = 0.7854$

As an approximation, use linear relationship for the values in Table 8.1. When $\beta = 0.1$, $\alpha_1 = 0.3113$

when $\beta = 1.0$, $\alpha_1 = 0.8602$

This gives $\alpha_1(\beta) = a + b\beta \equiv 0.6099\,\beta + 0.2503$

For $\alpha_1 = 0.7854$, we get $\beta = \dfrac{0.7854 - 0.2503}{0.6099} = 0.8774 = \dfrac{m}{M}$

$\therefore$ Mass to be attached $= M \simeq \dfrac{m}{0.8774} = 1.1397\, m$.

(8.17) Equation of motion for free vibration $\quad c^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$ $\quad (E_1)$

Assume separation of variables in Eq. (E_1): $\quad u(x,t) = U(x)\cdot T(t)$

so that $\quad c^2\dfrac{d^2U}{dx^2}T = U\dfrac{d^2T}{dt^2} \quad$ or $\quad c^2\dfrac{1}{U}\dfrac{d^2U}{dx^2} = \dfrac{1}{T}\dfrac{d^2T}{dt^2} = \alpha = -\omega^2$

--- (E_2)

i.e., $$c^2 \frac{d^2U(x)}{dx^2} + \omega^2 U(x) = 0 \qquad (E_3)$$

and $$\frac{d^2T(t)}{dt^2} + \omega^2 T(t) = 0 \qquad (E_4)$$

To show that the constant a on the right hand side of Eq. (E_2) is a negative quantity, multiply Eq. (E_3) by $U(x)$ and integrate w.r.t. x from 0 to l:

$$c^2 \int_0^l U(x) \frac{d^2U}{dx^2}\, dx = a \underbrace{\int_0^l \{U(x)\}^2\, dx}_{\text{positive always}} \qquad (E_5)$$

Eq. (E_5) shows that a will have the same sign as the integral on the left side:

$$I = \int_0^l U(x) \frac{d^2U(x)}{dx^2}\, dx \qquad (E_6)$$

Integrate (E_6) by parts to get

$$I = U(l) \cdot \frac{dU(l)}{dx} - U(0) \cdot \frac{dU(0)}{dx} - \int_0^l \left\{ \frac{dU(x)}{dx} \right\}^2 dx \qquad (E_7)$$

Since $U(0) = 0$, the second term on the r.h.s. of (E_7) will be zero. Also, $AE \frac{dU(l)}{dx} = -k\, U(l)$ $\qquad (E_8)$

at $x = l$. Multiplication of (E_8) by $U(l)$ gives

$$U(l) \cdot \frac{dU}{dx}(l) = -k \{U(l)\}^2 < 0$$

This shows that $I < 0$ and hence $a < 0$.

In Eqs. (E_3) and (E_4), there exists an eigen (normal) function for each frequency (constant) ω. Let $U_m(x)$ and $U_n(x)$ denote the normal functions corresponding to the frequencies ω_m and ω_n, respectively. Eq. (E_3) gives

$$c^2 \frac{d^2U_m}{dx^2} + \omega_m^2 U_m = 0 \;---\; (E_9)\;; \qquad c^2 \frac{d^2U_n}{dx^2} + \omega_n^2 U_n = 0 \;---\; (E_{10})$$

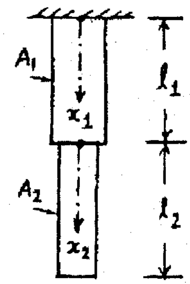
Multiply (E_9) by U_n, (E_{10}) by U_m and subtract the resulting equations one from the other to get

$$c^2 U_n \frac{d^2U_m}{dx^2} - c^2 U_m \frac{d^2U_n}{dx^2} + (\omega_m^2 - \omega_n^2) U_m U_n = 0 \qquad (E_{11})$$

Integrate (E_{11}) with respect to x from 0 to l to get, after integration by parts, the desired orthogonality relation:

$$\int_0^l U_m(x)\, U_n(x)\, dx = 0 \quad , \quad \omega_m \neq \omega_n \qquad (E_{12})$$

8.18 Set up two coordinates x_1 and x_2 as shown.



$$u_1(x_1,t) = \left(\underset{\sim}{A_1} \cos\frac{\omega x_1}{c_1} + \underset{\sim}{B_1} \sin\frac{\omega x_1}{c_1}\right)(C\cos\omega t + D\sin\omega t)$$

$$u_2(x_2,t) = \left(\underset{\sim}{A_2} \cos\frac{\omega x_2}{c_2} + \underset{\sim}{B_2} \sin\frac{\omega x_2}{c_2}\right)(C\cos\omega t + D\sin\omega t)$$

$$u_1(0,t) = 0 \quad\Rightarrow\quad \underset{\sim}{A_1} = 0$$

$$u_1(\ell_1,t) = u_2(0,t) \quad\Rightarrow\quad \underset{\sim}{B_1} \sin\frac{\omega \ell_1}{c_1} = \underset{\sim}{A_2}$$

$$A_1 E_1 \frac{\partial u_1}{\partial x_1}(\ell_1,t) = A_2 E_2 \frac{\partial u_2}{\partial x_2}(0,t) = \text{tensile force same in both areas}$$

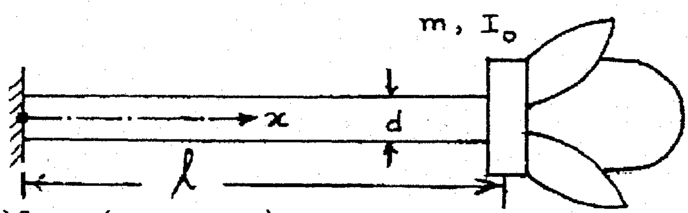
i.e. $$A_1 E_1 \underset{\sim}{B_1} \frac{\omega}{c_1} \cos\frac{\omega \ell_1}{c_1} = A_2 E_2 \frac{\omega}{c_2} \underset{\sim}{B_2} \quad\Rightarrow\quad \underset{\sim}{B_2} = \frac{A_1 E_1 c_2}{A_2 E_2 c_1} \cos\frac{\omega \ell_1}{c_1} \cdot \underset{\sim}{B_1}$$

$$u_2(x_2,t) = \underset{\sim}{B_1}\left(\sin\frac{\omega \ell_1}{c_1} \cos\frac{\omega x_2}{c_2} + \frac{A_1 E_1 c_2}{A_2 E_2 c_1} \cos\frac{\omega \ell_1}{c_1} \sin\frac{\omega x_2}{c_2}\right)(C\cos\omega t + D\sin\omega t)$$

$$\frac{\partial u_2}{\partial x_2}(\ell_2,t) = 0 \quad\Rightarrow\quad \underset{\sim}{B_1} \frac{\omega}{c_2}\left\{-\sin\frac{\omega \ell_1}{c_1} \cdot \sin\frac{\omega \ell_2}{c_2} + \frac{A_1 E_1 c_2}{A_2 E_2 c_1} \cdot \cos\frac{\omega \ell_1}{c_1} \cdot \cos\frac{\omega \ell_2}{c_2}\right\} = 0$$

$\therefore$ Frequency equation is
$$\tan\frac{\omega \ell_1}{c_1} \cdot \tan\frac{\omega \ell_2}{c_2} = \frac{A_1 E_1 c_2}{A_2 E_2 c_1}$$

8.19 (a) Axial vibration:



$$\beta = \frac{m_0}{m} = \frac{\text{mass of rod}}{\text{end mass}}$$

Using $\rho = 76.5\ \text{kN/m}^3$ for steel, we find

$$m_0 = \frac{\pi\, d^2\, \ell\, \rho}{4} = \frac{\pi}{4}\left(\frac{5}{100}\right)^2 (1) \left(\frac{76.5\ (10^3)}{9.81}\right) = 15.3117\ \text{kg}$$

$$\frac{m_0}{m} = \frac{15.3117}{100} = 0.1531$$

From Table 8.1, the value of α_1 for $\beta = 0.1531$ (using linear interpolation between values of $\beta = 0.1$ and $\beta = 1.0$) is:

$$\alpha_1 = 0.6099\ (0.1531) + 0.2503 = 0.3437$$

$$\omega_1 = \frac{\alpha_1\, c}{\ell} = \frac{\alpha_1}{\ell}\sqrt{\frac{E}{\rho}} = \frac{0.3437}{1}\sqrt{\frac{207\ (10^9)\ (9.81)}{76500}} = 1770.7958\ \text{rad/sec}$$

(b) Torsional vibration:

In this case, we use the result of Example 8.6.

$$\beta = \frac{\bar{J}_{rod}}{I_0} = \frac{J\,\rho\,\ell}{I_0}$$

where J = polar moment of inertia of the shaft:

$$J = \frac{\pi}{32} d^4 = \frac{\pi}{32} (0.05)^4 = 2.4544\ (10^{-8})\ m^4$$

$$\rho = 76500/9.81\ kg/m^3 \ ; \ \ell = 1\ m$$

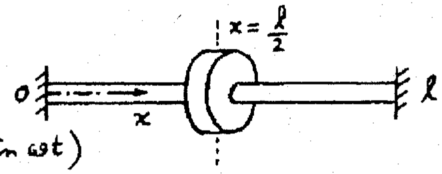
$$\tilde{J}_{rod} = (2.4544\ (10^{-8}))\ \frac{76500}{9.81}\ (1) = 1.9140\ (10^{-4})\ kg\text{-}m^2$$

$$\beta = \frac{1.9140\ (10^{-4})}{10} = 1.9140\ (10^{-5})$$

Since β is close to zero, we have from Example 8.6,

$$\omega_1 \approx \frac{c}{\ell} \sqrt{\beta} = \sqrt{\frac{G\,\beta}{\rho}} = \sqrt{\frac{(80\ (10^9))\ (9.81)}{76500}} = 14.0126\ \text{rad/sec}$$

8.20 Consider half the shaft and half the disc.



$$\theta(x,t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right)$$

$$\theta(0,t) = 0 \quad \Rightarrow \quad A = 0$$

$$GJ \frac{\partial \theta}{\partial x}\left(\frac{\ell}{2}, t\right) = -\frac{J_0}{2} \frac{\partial^2 \theta}{\partial t^2}\left(\frac{\ell}{2}, t\right)$$

$$\Rightarrow \quad GJ \frac{\omega}{c} \cdot B \cos \frac{\omega \ell}{2c} = \frac{J_0\, \omega^2}{2} \cdot B \sin \frac{\omega \ell}{2c}$$

$$\Rightarrow \quad \tan \frac{\omega \ell}{2c} = \frac{2GJ}{J_0\, \omega c} = \frac{2c}{\omega \ell} \cdot \frac{GJ\ell}{J_0 c^2}$$

Frequency equation: $\alpha \tan \alpha = \beta$ where $\alpha = \frac{\omega \ell}{2c}$ and $\beta = \frac{GJ\ell}{J_0 c^2}$ $= \left(\frac{J\rho\ell}{J_0}\right)$

If roots of this equation are given by α_n,

$\omega_n = \frac{2\,\alpha_n c}{\ell}$ and

$$\theta(x,t) = \sum_{n=1}^{\infty} \sin \frac{\omega_n x}{c} \left(C_n \cos \omega_n t + D_n \sin \omega_n t\right) \quad \text{---- } (E_1)$$

$$\theta(x,0) = \sum_{n=1}^{\infty} \sin \frac{\omega_n x}{c} \left(C_n\right) = 0 \quad \Rightarrow \quad C_n = 0 \quad \text{---- } (E_2)$$

$$\dot{\theta}(x,0) = \sum_{n=1}^{\infty} \sin \frac{\omega_n x}{c} \left(\omega_n D_n\right) = \frac{2\,\dot{\theta}_0\, x}{\ell} \quad \text{---- } (E_3)$$

as $\dot{\theta}\Big|_{\frac{\ell}{2},\, t=0} = \dot{\theta}_0$ and hence $\dot{\theta}(x,0) = \frac{2\,\dot{\theta}_0\, x}{\ell}$

Multiply Eq. (E_3) by $\sin \frac{\omega_n x}{c}$ and integrate from 0 to $\frac{\pi c}{2\omega_n}$:

$$\omega_n D_n \int_0^{\left(\frac{\pi c}{2\omega_n}\right)} \sin^2 \frac{\omega_n x}{c}\, dx = \frac{2\,\dot{\theta}_0}{\ell} \int_0^{\left(\frac{\pi c}{2\omega_n}\right)} x \sin \frac{\omega_n x}{c}\, dx \quad (E_4)$$

i.e., $$D_n = \frac{8\, c\, \dot{\theta}_0}{\pi\, \ell\, \omega_n^2} \quad (E_5)$$

$$\therefore \quad \theta(x,t) = \sum_{n=1}^{\infty} \left(\frac{8\, c\, \dot{\theta}_0}{\pi l\, \omega_n^2} \right) \sin \frac{\omega_n x}{c} \, \sin \omega_n t \qquad (E_6)$$

8.21

$$\theta(x,t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right)$$

$$\theta(0,t) = 0 \Rightarrow A = 0$$

$$\theta(l,t) = 0 \Rightarrow \sin \frac{\omega l}{c} = 0 \quad ; \quad \frac{\omega_n l}{c} = n\pi$$

$$\therefore \omega_n = \frac{n\pi c}{l} = \frac{n\pi}{l}\sqrt{\frac{G}{\rho}} \quad ; \quad n = 1, 2, 3, \ldots$$

8.22

Boundary conditions:

At $x = 0$,
$$GJ \frac{\partial \theta}{\partial x}(0,t) = k_{t1}\, \theta(0,t) + c_{t1} \frac{\partial \theta}{\partial t}(0,t) + J_1 \frac{\partial^2 \theta}{\partial t^2}(0,t)$$

At $x = l$,
$$GJ \frac{\partial \theta}{\partial x}(l,t) = -k_{t2}\, \theta(l,t) - c_{t2} \frac{\partial \theta}{\partial t}(l,t) - J_2 \frac{\partial^2 \theta}{\partial t^2}(l,t)$$

8.23

$$\theta(x,t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right)$$

$$\theta(0,t) = 0 \Rightarrow A = 0$$

$$\frac{\partial \theta}{\partial x}(l,t) = 0 \Rightarrow \cos \frac{\omega l}{c} = 0 \quad ; \quad \frac{\omega_n l}{c} = \left(n - \tfrac{1}{2}\right)\pi$$

$$\therefore \omega_n = \left(n - \tfrac{1}{2}\right)\pi \sqrt{\frac{G}{\rho l^2}} \quad ; \quad n = 1, 2, \ldots$$

8.24

$$\theta(x,t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right) \quad \text{----} \; (E_1)$$

at $x = 0$:
$$J_1 \frac{\partial^2 \theta}{\partial t^2} = GJ \frac{\partial \theta}{\partial x} \quad \text{----} \; (E_2)$$

at $x = l$:
$$-J_2 \frac{\partial^2 \theta}{\partial t^2} = GJ \frac{\partial \theta}{\partial x} \quad \text{----} \; (E_3)$$

Eqs. (E_1) and (E_2) give
$$(J_1 \omega^2) A + \left(\frac{GJ\omega}{c}\right) B = 0 \quad \text{----} \; (E_4)$$

(E_1) and (E_3) give
$$\left(J_2 \omega^2 \cos \frac{\omega l}{c} + \frac{GJ\omega}{c} \sin \frac{\omega l}{c}\right) A + \left(J_2 \omega^2 \sin \frac{\omega l}{c} - \frac{GJ\omega}{c} \cos \frac{\omega l}{c}\right) B = 0 \quad \text{----} \; (E_5)$$

Setting the determinant of the coefficients of A and B in (E_4) and (E_5) equal to zero, we get the frequency equation

$$\begin{vmatrix} J_1 \omega^2 & \dfrac{GJ\omega}{c} \\ \left(J_2 \omega^2 \cos \dfrac{\omega l}{c} + \dfrac{GJ\omega}{c} \sin \dfrac{\omega l}{c}\right) & \left(J_2 \omega^2 \sin \dfrac{\omega l}{c} - \dfrac{GJ\omega}{c} \cos \dfrac{\omega l}{c}\right) \end{vmatrix} = 0.$$

i.e.
$$J_1 J_2 \omega^2 \sin \frac{\omega l}{c} - (J_1 + J_2) \frac{GJ\omega}{c} \cos \frac{\omega l}{c} - \frac{G^2 J^2}{c^2} \sin \frac{\omega l}{c} = 0$$

8.25 Equation of motion: $c^2 \dfrac{\partial^2 \theta}{\partial x^2} = \dfrac{\partial^2 \theta}{\partial t^2}$ ---- (E_1)

Let steady state vibration be $\theta(x,t) = X(x) \cdot \cos \omega t$ ---- (E_2)

(E_1) and (E_2) give $\dfrac{d^2 X}{dx^2} + \dfrac{\omega^2}{c^2} X = 0$

Solution is $X(x) = A \cos \dfrac{\omega x}{c} + B \sin \dfrac{\omega x}{c}$

$\theta(x=0, t) = 0 \Rightarrow X(x=0) = 0 \Rightarrow A = 0$

At free end, $G J \dfrac{\partial \theta}{\partial x}(x = l, t) = M_{t0} \cos \omega t$

$$B \, G J \frac{\omega}{c} \cos \frac{\omega l}{c} \cos \omega t = M_{t0} \cos \omega t$$

$$B = \frac{M_{t0}\, c}{G J \omega} \sec \frac{\omega l}{c}$$

$$\therefore \quad \theta(x,t) = \frac{M_{t0}\, c}{G J \omega} \sec \frac{\omega l}{c} \sin \frac{\omega x}{c} \cdot \cos \omega t$$

8.26 From solution of problem 8.21, $\omega_n = \dfrac{n\pi}{l} \sqrt{\dfrac{G}{\rho}}$

$\therefore \omega_1 = \dfrac{\pi}{l} \sqrt{\dfrac{G}{\rho}} = \dfrac{\pi}{2} \sqrt{\dfrac{0.8 \times 10^{11}}{7800}} = 5030.5861$ rad/sec

---

8.27 Let the angular displacement of the shaft be measured from the position occupied at the instant the shaft is stopped. Then the initial conditions are

$\theta(x,0) = 0, \quad \dfrac{\partial \theta}{\partial t}(x,0) = \underset{\sim}{\omega}$ ---- (E_1)

Angular displacement of shaft is

$\theta(x,t) = \left(A \cos \dfrac{\omega x}{c} + B \sin \dfrac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right)$ --- (E_2)

As the shaft is supported at $x=0$ and free at $x = l$,

$\theta(0,t) = 0, \quad \dfrac{\partial \theta}{\partial x}(l,t) = 0$ ---- (E_3)

Eqs. (E_2) and (E_3) give $A = 0$

$B \dfrac{\omega}{c} \cos \dfrac{\omega l}{c} = 0$ or $\dfrac{\omega_n l}{c} = \dfrac{n\pi}{2}$; $n = 1, 3, \dots$

$\theta_n(x,t) = \sin \dfrac{n\pi x}{2l}\left(C_n \cos \omega_n t + D_n \sin \omega_n t\right)$ ---- (E_4)

$\theta(x,t) = \displaystyle\sum_{n=1,3,\dots}^{\infty} \sin \dfrac{n\pi x}{2l}\left(C_n \cos \omega_n t + D_n \sin \omega_n t\right)$ ---- (E_5)

At $t=0$, $\theta(x,0) = \displaystyle\sum_{n=1,3,\dots}^{\infty} \sin \dfrac{n\pi x}{2l} \, C_n$ ---- (E_6)

$\dfrac{\partial \theta}{\partial t}(x,0) = \displaystyle\sum_{n=1,3,\dots}^{\infty} \sin \dfrac{n\pi x}{2l} \, \omega_n D_n = \underset{\sim}{\omega}$ ---- (E_7)

Since $\theta(x,0) = 0$, $C_n = 0$. By multiplying both sides of (E_7) by $\sin \dfrac{n\pi x}{2l}$, integrating from 0 to l, and noting that

$$\int_0^{l} \sin^2 \frac{n\pi x}{2l}\, dx = \frac{l}{2} \quad \text{and} \quad \int_0^{l} \underset{\sim}{\omega} \sin \frac{n\pi x}{2l}\, dx = \underset{\sim}{\omega}\, \frac{2l}{n\pi},$$

we get $\quad \underset{\sim}{\omega}\, \dfrac{2l}{n\pi} = \dfrac{\omega_n l}{2} D_n = \dfrac{n\pi c}{4} D_n \Rightarrow D_n = \dfrac{8 l \underset{\sim}{\omega}}{n^2 \pi^2 c}\ ; \ n = 1, 3, \ldots$

$$\theta(x,t) = \sum_{n=1,3,\ldots}^{\infty} \sin \frac{n\pi x}{2l} \left(\frac{8 l \underset{\sim}{\omega}}{n^2 \pi^2 c} \right) \sin \omega_n t$$

8.28 $I = \frac{1}{12}(0.1)(0.3)^3 = 2.25 \times 10^{-4}\ m^4$

$A = 0.03\ m^2, \quad l = 2\ m, \quad E = 20.5 \times 10^{10}\ N/m^2, \quad \rho = 7.83 \times 10^3\ kg/m^3$

Fig. 8.15 gives the values of $\beta_n l$:

$$\omega_n = (\beta_n l)^2 \sqrt{\frac{E I}{\rho A l^4}}$$

Here $\sqrt{\dfrac{E I}{\rho A l^4}} = \left\{ \dfrac{(20.5 \times 10^{10})(2.25 \times 10^{-4})}{7830 \times 0.03 \times 16} \right\}^{\frac{1}{2}} = 110.7814$

(a) For pinned-pinned beam:

$\beta_1 l = \pi, \quad \omega_1 = \pi^2 (110.7814) = 1093.3737$ rad/sec

$\beta_2 l = 2\pi, \quad \omega_2 = 4\,\omega_1 = 4373.4948$ rad/sec

$\beta_3 l = 3\pi, \quad \omega_3 = 9\,\omega_1 = 9840.3634$ rad/sec

(b) For fixed-fixed beam:

$\beta_1 l = 4.730841, \quad \omega_1 = 2479.3826$ rad/sec

$\beta_2 l = 7.853205, \quad \omega_2 = 6832.2023$ rad/sec

$\beta_3 l = 10.995608, \quad \omega_3 = 13393.8474$ rad/sec

(c) For fixed-free beam:

$\beta_1 l = 1.875104, \quad \omega_1 = 389.5091$ rad/sec

$\beta_2 l = 4.694091, \quad \omega_2 = 2441.0117$ rad/sec

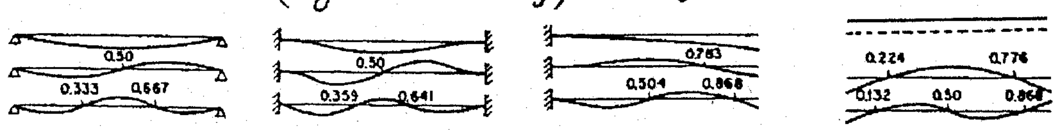
$\beta_3 l = 7.854757, \quad \omega_3 = 6834.9030$ rad/sec

(d) For free-free beam:

$\beta_1 l = 0, \quad \omega_1 = 0; \quad \beta_2 l = 4.730841, \quad \omega_2 = 2479.3826$ rad/sec

$\beta_3 l = 7.853205, \quad \omega_3 = 6832.2023$ rad/sec.

Mode shapes: The mode shapes are given in Fig. 8.15 (equations only). They result in the following.



Pinned-pinned Fixed-fixed Fixed-free Free-free

8.29 Boundary conditions are:

At $x=0$, fixed end $\Rightarrow$ $W(0)=0$ --- (E_1) , $\frac{dW}{dx}(0)=0$ --- (E_2)

At $x=l$, free end $\Rightarrow$ $\frac{d^2W}{dx^2}(l)=0$ --- (E_3) , $\frac{d^3W}{dx^3}(0)=0$ --- (E_4)

The deflection (normal) function is given by

$$W(x) = C_1 \cos\beta x + C_2 \sin\beta x + C_3 \cosh\beta x + C_4 \sinh\beta x \qquad (E_5)$$

from which

$$\frac{dW}{dx}(x) = \beta\left[-C_1 \sin\beta x + C_2 \cos\beta x + C_3 \sinh\beta x + C_4 \cosh\beta x\right] \qquad (E_6)$$

Eqs. (E_1) and (E_5) give $\quad C_1 + C_3 = 0 \qquad (E_7)$

Eqs. (E_2) and (E_6) yield $\quad \beta\,(C_2 + C_4) = 0 \qquad (E_8)$

Use of (E_7) and (E_8) in (E_5) leads to

$$W(x) = C_1(\cos\beta x - \cosh\beta x) + C_2(\sin\beta x - \sinh\beta x) \qquad (E_9)$$

Use of Eqs. (E_3), (E_4) and (E_9) yields

$$C_1(\cos\beta l + \cosh\beta l) + C_2(\sin\beta l + \sinh\beta l) = 0 \qquad (E_{10})$$

$$C_1(\sin\beta l - \sinh\beta l) - C_2(\cos\beta l + \cosh\beta l) = 0 \qquad (E_{11})$$

The frequency equation can be obtained by setting the coefficient matrix in (E_{10}) and (E_{11}) to zero as

$$\begin{vmatrix} \cos\beta l + \cosh\beta l & \sin\beta l + \sinh\beta l \\ \sin\beta l - \sinh\beta l & -\cos\beta l - \cosh\beta l \end{vmatrix} = 0 \qquad (E_{12})$$

Upon simplification, Eq. (E_{12}) yields the frequency equation

$$\cos\beta l \cdot \cosh\beta l = -1 \qquad (E_{13})$$

First four roots of (E_{13}) are given by

$\beta_1 l = 1.875104$, $\beta_2 l = 4.694091$, $\beta_3 l = 7.854757$, $\beta_4 l = 10.995541$.

(8.30) Boundary conditions are:

At $x=0$; $\quad EI\,\frac{d^3W}{dx^3}=0$ --- (E_1), $\quad EI\,\frac{d^2W}{dx^2}=-k_t\,\frac{dW}{dx}$ --- (E_2)

At $x=l$; $\quad EI\,\frac{d^3W}{dx^3}=kW$ --- (E_3), $\quad EI\,\frac{d^2W}{dx^2}=0$ --- (E_4)

Eq. (8.105) gives

$$\int_0^l W_i\,W_j\,dx=-\frac{c^2}{\omega_i^2-\omega_j^2}\left[W_i\,W_j'''-W_j\,W_i'''+W_j'\,W_i''-W_i'\,W_j''\right]_0^l \quad \text{--- } (E_5)$$

At $x=l$, Eq. (E_4) gives $\quad W_i''=0$ --- (E_6), $\quad W_j''=0$ --- (E_7)

Eq. (E_3) gives

$EI\,W_i'''=k\,W_i$ --- (E_8), $\quad EI\,W_j'''=k\,W_j$ --- (E_9)

Multiplying (E_8) by W_j, (E_9) by W_i, and subtracting one from the other, we get

$$EI\left(W_j'''\,W_i-W_i'''\,W_j\right)=k\left(W_j\,W_i-W_i\,W_j\right)=0 \quad \text{---} (E_{10})$$

This shows that the r.h.s. of (E_5) is zero at $x=l$.

At $x=0$, Eq. (E_1) gives $\quad W_i'''=0$ --- (E_{11}), $\quad W_j'''=0$ --- (E_{12})

Eq. (E_2) gives

$EI\,W_i''=-k_t\,W_i'$ --- (E_{13}), $\quad EI\,W_j''=-k_t\,W_j'$ --- (E_{14})

Multiplying (E_{13}) by W_j', (E_{14}) by W_i', and subtracting one from the other, we get

$$EI\left(W_i''\,W_j'-W_j''\,W_i'\right)=-k_t\left(W_i'\,W_j'-W_j'\,W_i'\right)=0 \quad \text{---} (E_{15})$$

This shows that the r.h.s. of Eq. (E_5) is zero at $x=0$.

(8.31) $w(0,t)=\frac{\partial^2 w}{\partial x^2}(0,t)=w(l,t)=\frac{\partial^2 w}{\partial x^2}(l,t)=0\,;\quad t\geq 0$

$$W(x)=C_1(\cos\beta x+\cosh\beta x)+C_2(\cos\beta x-\cosh\beta x)$$
$$+C_3(\sin\beta x+\sinh\beta x)+C_4(\sin\beta x-\sinh\beta x)$$

$$\frac{d^2W}{dx^2}(x)=C_1\,\beta^2(-\cos\beta x+\cosh\beta x)+C_2\,\beta^2(-\cos\beta x-\cosh\beta x)$$
$$+C_3\,\beta^2(-\sin\beta x+\sinh\beta x)+C_4\,\beta^2(-\sin\beta x-\sinh\beta x)$$

$W(0)=0 \Rightarrow \quad C_1=0$

$\frac{d^2W}{dx^2}(0)=0 \Rightarrow \quad C_2=0$

$W(l)=0 \Rightarrow \quad C_3(\sin\beta l+\sinh\beta l)+C_4(\sin\beta l-\sinh\beta l)=0$ -- (E_1)

$\frac{d^2W}{dx^2}(l)=0 \Rightarrow C_3(-\sin\beta l+\sinh\beta l)-C_4(\sin\beta l+\sinh\beta l)=0$ -- (E_2)

Setting the determinant of the coefficient matrix of C_3 and C_4 in (E_1) and (E_2) to zero, we get the frequency equation

$$-(\sin\beta l + \sinh\beta l)^2 + (\sin\beta l - \sinh\beta l)^2 = 0$$

or $$\sin\beta l \, \sinh\beta l = 0$$

Since $\sinh\beta l \neq 0$, the frequency equation becomes

$$\sin\beta l = 0$$

$$\therefore \beta_n l = n\pi \quad ; \quad \omega_n = \frac{n^2\pi^2}{l^2}\sqrt{\frac{EI}{\rho A}} \quad ; n = 1, 2, \ldots$$

8.32 $$w(0,t) = \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(l,t) = \frac{\partial^3 w}{\partial x^3}(l,t) = 0 \ ; \quad t \geq 0$$

$$W(x) = C_1(\cos\beta x + \cosh\beta x) + C_2(\cos\beta x - \cosh\beta x) + C_3(\sin\beta x + \sinh\beta x) + C_4(\sin\beta x - \sinh\beta x)$$

$$\frac{d^2W}{dx^2}(x) = C_1\beta^2(-\cos\beta x + \cosh\beta x) + C_2\beta^2(-\cos\beta x - \cosh\beta x) + C_3\beta^2(-\sin\beta x + \sinh\beta x) + C_4\beta^2(-\sin\beta x - \sinh\beta x)$$

$$\frac{d^3W}{dx^3}(x) = C_1\beta^3(\sin\beta x + \sinh\beta x) + C_2\beta^3(\sin\beta x - \sinh\beta x) + C_3\beta^3(-\cos\beta x + \cosh\beta x) + C_4\beta^3(-\cos\beta x - \cosh\beta x)$$

$$W(0) = 0 \Rightarrow C_1 = 0$$

$$\frac{d^2W}{dx^2}(0) = 0 \Rightarrow C_2 = 0$$

$$\frac{d^2W}{dx^2}(l) = 0 \Rightarrow C_3(-\sin\beta l + \sinh\beta l) - C_4(\sin\beta l + \sinh\beta l) = 0 \quad \text{---} (E_1)$$

$$\frac{d^3W}{dx^3}(l) = 0 \Rightarrow C_3(-\cos\beta l + \cosh\beta l) - C_4(\cos\beta l + \cosh\beta l) = 0 \quad \text{---} (E_2)$$

Setting the determinant of coefficients of C_3 and C_4 in Eqs. (E_1) and (E_2) to zero, the frequency equation can be obtained:

$$-(-\sin\beta l + \sinh\beta l)(\cos\beta l + \cosh\beta l) + (\sin\beta l + \sinh\beta l)(-\cos\beta l + \cosh\beta l) = 0$$

or $$\tan\beta l - \tanh\beta l = 0$$

8.33 For a simply supported beam, the first three natural frequencies are given by

$$\omega_1 = (\beta_1 l)^2 \left(EI/\rho A l^4\right)^{\frac{1}{2}} = \pi^2 \left(EI/\rho A l^4\right)^{1/2}$$

$$\omega_2 = (\beta_2 l)^2 \left(EI/\rho A l^4\right)^{1/2} = 4\pi^2 \left(EI/\rho A l^4\right)^{\frac{1}{2}}$$

$$\omega_3 = (\beta_3 l)^2 \left(EI/\rho A l^4\right)^{\frac{1}{2}} = 9\pi^2 \left(EI/\rho A l^4\right)^{1/2}$$

For steel, $E = 2.07 \times 10^{11}$ N/m², $\rho = 7880$ kg/m³, $l = 1.0$ m.

Setting $\omega_1 \geq 1500$ Hz $= 9424.8$ rad/sec $= \pi^2 \left\{ \frac{(2.07 \times 10^{11}) I}{(7880) A (1)} \right\}^{1/2}$

we get

$$(9424.8)^2 \geq \pi^4 \left(\frac{2.07 \times 10^{11}}{7880} \right) \left(\frac{I}{A} \right)$$

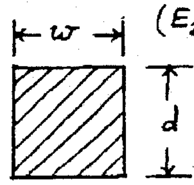
or

$$\frac{I}{A} \leq 0.03471 \qquad (E_1)$$

Setting $\omega_3 \leq 5000$ Hz $= 31416.0$ rad/sec $= 9\pi^2 \left\{ \frac{(2.07 \times 10^{11}) I}{7880 A (1)} \right\}^{\frac{1}{2}}$

we get

$$\frac{I}{A} \geq 0.0047617 \qquad (E_2)$$



Let the beam have a rectangular section

$$A = w d, \quad I = \frac{1}{12} w d^3, \quad \frac{I}{A} = \frac{d^2}{12}$$

Let $\frac{I}{A} = 0.005$ to satisfy inequalities (E_1) and (E_2).

Then $d^2 = 0.06$ or $d = 0.2449$ m

Taking $w = 0.1$ m (w can have any value), we get

$$A = w d = 0.02449 \text{ m}^2 \quad \text{and} \quad I = 1.2240 \times 10^{-4} \text{ m}^4.$$

8.34 From the solution of problem 8.31, we find for $\sin \beta l = 0$,

$$C_3 = C_4$$

Hence $W_n(x) = C_n \sin \beta_n x = C_n \sin \frac{n \pi x}{l}$

General free vibration solution is

$$w(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

When beam vibrates in the first mode,

$$w(x,t) = \sin \frac{\pi x}{l} (A_1 \cos \omega_1 t + B_1 \sin \omega_1 t)$$

Bending moment in the beam is

$$M(x,t) = EI \frac{\partial^2 w}{\partial x^2}(x,t) = -\frac{\pi^2 EI}{l^2} \sin \frac{\pi x}{l} (A_1 \cos \omega_1 t + B_1 \sin \omega_1 t)$$

$$= -\frac{\pi^2}{l^2} EI \, w(x,t)$$

If the amplitude of vibration is Y, the maximum bending moment is

$$|M_{max}| = \frac{\pi^2}{l^2} EI \, Y$$

For the given data,

$$|M_{max}| = \frac{\pi^2}{(1)^2} (20.5 \times 10^{10}) \left(\frac{10^3}{10^{12}} \right) \left(\frac{10}{1000} \right) = 20.2328 \text{ N-m}$$

8.35

$$W(x) = C_1 \cos\beta x + C_2 \sin\beta x + C_3 \cosh\beta x + C_4 \sinh\beta x$$

$$\frac{d^2W}{dx^2}(x) = -C_1\beta^2\cos\beta x - C_2\beta^2\sin\beta x + C_3\beta^2\cosh\beta x + C_4\beta^2\sinh\beta x$$

$$\frac{d^3W}{dx^3}(x) = C_1\beta^3\sin\beta x - C_2\beta^3\cos\beta x + C_3\beta^3\sinh\beta x + C_4\beta^3\cosh\beta x$$

$$EI\,\frac{d^2W}{dx^2}(0) = 0 \Rightarrow \quad C_1 = C_3 \qquad \text{--- } (E_1)$$

$$EI\,\frac{d^3W}{dx^3}(0) = -\underset{\sim}{k}_1\, W(0) \Rightarrow \quad EI\beta^3(-C_2+C_4) + \underset{\sim}{k}_1(C_1+C_3) = 0$$

$$C_2(-EI\beta^3) + C_3(+2\underset{\sim}{k}_1) + C_4(EI\beta^3) = 0 \qquad \text{--- } (E_2)$$

$$EI\,\frac{d^2W}{dx^2}(l) = 0 \Rightarrow C_2(-\sin\beta l) + C_3(\cosh\beta l - \cos\beta l) + C_4(\sinh\beta l) = 0 \qquad \text{--- } (E_3)$$

$$EI\,\frac{d^3W}{dx^3}(l) = +\underset{\sim}{k}_2\, W(l) \Rightarrow$$

$$C_2(-EI\beta^3\cos\beta l - \underset{\sim}{k}_2\sin\beta l) + C_3(EI\beta^3\sin\beta l + EI\beta^3\sinh\beta l - \underset{\sim}{k}_2\cos\beta l - \underset{\sim}{k}_2\cosh\beta l) + C_4(EI\beta^3\cosh\beta l - \underset{\sim}{k}_2\sinh\beta l) = 0 \qquad \text{--- } (E_4)$$

Setting the determinant of the coefficients of C_2, C_3 and C_4 in (E_2) to (E_4) to zero, we get the frequency equation:

$$\begin{vmatrix} -EI\beta^3 & +2\underset{\sim}{k}_1 & EI\beta^3 \\ -\sin\beta l & (\cosh\beta l - \cos\beta l) & \sinh\beta l \\ (-EI\beta^3\cos\beta l - \underset{\sim}{k}_2\sin\beta l) & (EI\beta^3\{\sin\beta l + \sinh\beta l\} - \underset{\sim}{k}_2\{\cos\beta l + \cosh\beta l\}) & (EI\beta^3\cosh\beta l - \underset{\sim}{k}_2\sinh\beta l) \end{vmatrix} = 0$$

8.36

Because of symmetry, consider only half of beam to the left of M. Boundary conditions are:

$$w(0,t) = 0, \quad EI\,\frac{\partial^2 w}{\partial x^2}(0,t) = 0$$

For symmetry, $\frac{\partial w}{\partial x} = 0$ at $x = \frac{l}{2}$.

$$EI\,\frac{\partial^3 w}{\partial x^3} = +\frac{M}{2}\,\frac{\partial^2 w}{\partial t^2} \quad \text{at } x = \frac{l}{2}$$

$$W(x) = C_1(\cos\beta x + \cosh\beta x) + C_2(\cos\beta x - \cosh\beta x) + C_3(\sin\beta x + \sinh\beta x) + C_4(\sin\beta x - \sinh\beta x)$$

$$W(0) = 0 \Rightarrow C_1 = 0$$

$$\frac{d^2W}{dx^2}(0) = 0 \Rightarrow C_2 = 0$$

$$W(x) = C_3(\sin\beta x + \sinh\beta x) + C_4(\sin\beta x - \sinh\beta x)$$

$$\frac{dW}{dx}(x) = \beta\, C_3\,(\cos\beta x + \cosh\beta x) + \beta C_4\,(\cos\beta x - \cosh\beta x)$$

$$\frac{d^3W}{dx^3}(x) = \beta^3 C_3\,(-\cos\beta x + \cosh\beta x) - \beta^3 C_4\,(\cos\beta x + \cosh\beta x)$$

$$\frac{dW}{dx}\left(\frac{l}{2}\right) = 0 \Rightarrow \quad C_3\left(\cos\frac{\beta l}{2} + \cosh\frac{\beta l}{2}\right) + C_4\left(\cos\frac{\beta l}{2} - \cosh\frac{\beta l}{2}\right) = 0 \quad \text{----} \; (E_1)$$

$$EI\,\frac{d^3W}{dx^3}\left(\frac{l}{2}\right) = -\frac{M}{2}\,\omega^2\, W\left(\frac{l}{2}\right) \quad \Rightarrow$$

$$EI\,\beta^3\left[C_3\left(-\cos\frac{\beta l}{2} + \cosh\frac{\beta l}{2}\right) - C_4\left(\cos\frac{\beta l}{2} + \cosh\frac{\beta l}{2}\right)\right] =$$

$$-\frac{M}{2}\,\omega^2\left[C_3\left(\sin\frac{\beta l}{2} + \sinh\frac{\beta l}{2}\right) + C_4\left(\sin\frac{\beta l}{2} - \sinh\frac{\beta l}{2}\right)\right] \text{---} (E_2)$$

i.e.

$$R_1 \cos\frac{\beta l}{2} + R_2\cosh\frac{\beta l}{2} = 0$$

$$R_1\left(-\cos\frac{\beta l}{2} + \lambda\sin\frac{\beta l}{2}\right) + R_2\left(\cosh\frac{\beta l}{2} + \lambda\sinh\frac{\beta l}{2}\right) = 0$$

where $R_1 = C_3 + C_4$, $R_2 = C_3 - C_4$, $\lambda = \dfrac{M\omega^2}{2EI\beta^3}$.

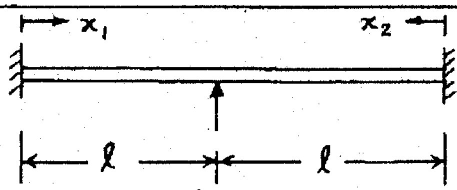
By setting the coefficient matrix of R_1 and R_2 equal to zero, we obtain the frequency equation:

$$\cos\frac{\beta l}{2}\left(\cosh\frac{\beta l}{2} + \lambda\sinh\frac{\beta l}{2}\right) + \cosh\frac{\beta l}{2}\left(\cos\frac{\beta l}{2} + \lambda\sin\frac{\beta l}{2}\right) = 0$$

or

$$\lambda\left(-\tanh\frac{\beta l}{2} + \tan\frac{\beta l}{2}\right) = 2$$

8.37 For convenience, set up two coordinates x_1 and x_2 as shown.



$$W_1(x_1) = C_1\cos\beta x_1 + C_2\sin\beta x_1 + C_3\cosh\beta x_1 + C_4\sinh\beta x_1$$

$$\frac{dW_1}{dx_1}(x_1) = -C_1\beta\sin\beta x_1 + C_2\beta\cos\beta x_1 + C_3\beta\sinh\beta x_1 + C_4\beta\cosh\beta x_1$$

$$W_1(0) = 0 \Rightarrow C_1 + C_3 = 0\;; \quad C_1 = -C_3$$

$$\frac{dW_1}{dx_1}(0) = 0 \Rightarrow C_2 + C_4 = 0\;; \quad C_2 = -C_4$$

$$W_1(x_1) = C_3(\cosh\beta x_1 - \cos\beta x_1) + C_4(\sinh\beta x_1 - \sin\beta x_1)$$

$$W_1(l) = 0 \Rightarrow \quad C_3 = -C_4\left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)$$

$$W_1(x_1) = C_4\left[(\sinh\beta x_1 - \sin\beta x_1) - \left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)(\cosh\beta x_1 - \cos\beta x_1)\right] \quad \text{----} \; (E_1)$$

$$W_2(x_2) = C_1'\cos\beta x_2 + C_2'\sin\beta x_2 + C_3'\cosh\beta x_2 + C_4'\sinh\beta x_2$$

$$\frac{dW_2}{dx_2}(x_2) = -c_1' \beta \sin\beta x_2 + c_2' \beta \cos\beta x_2 + c_3' \beta \sinh\beta x_2 + c_4' \beta \cosh\beta x_2$$

$$W_2(0) = 0 \Rightarrow \quad c_1' = -c_3'$$

$$\frac{dW_2}{dx_2}(0) = 0 \Rightarrow c_2' = -c_4'$$

$$W_2(l) = 0 \Rightarrow \quad c_3' = -c_4' \left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)$$

$$W_2(x_2) = c_4' \left[(\sinh\beta x_2 - \sin\beta x_2) - \left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)(\cosh\beta x_2 - \cos\beta x_2)\right] \quad \text{---} (E_2)$$

$\frac{dW_1}{dx_1}(x_1 = l) = -\frac{dW_2}{dx_2}(x_2 = l)$ gives

$$C_4\left[(\cosh\beta l - \cos\beta l) - \left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)(\sinh\beta l + \sin\beta l)\right]$$
$$+ c_4'\left[(\cosh\beta l - \cos\beta l) - \left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)(\sinh\beta l + \sin\beta l)\right] = 0 \quad \text{---} (E_3)$$

$\frac{d^2W_1}{dx_1^2}(x_1 = l) = -\frac{d^2W_2}{dx_2^2}(x_2 = l)$ gives

$$C_4\left[(\sinh\beta l + \sin\beta l) - \left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)(\cosh\beta l + \cos\beta l)\right]$$
$$+ c_4'\left[(\sinh\beta l + \sin\beta l) - \left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)(\cosh\beta l + \cos\beta l)\right] = 0 \quad \text{---} (E_4)$$

Since the determinant of the coefficients of C_4 and c_4' in (E_3) and (E_4) is identically zero, we set the coefficients of C_4 and c_4' in (E_3) and (E_4) to zero. This leads to the frequency equations

$$\cos\beta l \cosh\beta l = 1$$

and $$\tan\beta l = \tanh\beta l$$

8.38 General free vibration solution of a simply supported beam is

$$w(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\,(A_n \cos\omega_n t + B_n \sin\omega_n t)$$

At $t=0$, $\frac{\partial w}{\partial t}(x,0) = 0$ and $EI\frac{\partial^4 w}{\partial x^4}(x,0) = f_0$

Thus $$\sum_{n=1}^{\infty} B_n \omega_n \sin\frac{n\pi x}{l} = 0 \quad \text{or} \quad B_n = 0$$

and $$EI\sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{l}\right)^4 \sin\frac{n\pi x}{l} = f_0 \quad \text{----} (E_1)$$

Multiplying Eq. (E_1) by $\sin \frac{n\pi x}{l}$ and integrating from 0 to l, we obtain

$$A_n = \frac{2\,l^3}{EI\, n^4 \pi^4} \int_0^l f_0 \sin \frac{n\pi x}{l}\, dx = \frac{2\, l^4 f_0}{EI\, n^5 \pi^5} (1 - \cos n\pi)$$

$$= \frac{4 f_0\, l^4}{EI\, n^5 \pi^5} \quad ; \; n = 1, 3, 5, \ldots$$

$\therefore$ Solution is $\quad w(x,t) = \dfrac{4 f_0\, l^4}{\pi^5 EI} \displaystyle\sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^5} \sin \frac{n\pi x}{l} \cos \omega_n t$

8.39 Let $W(x) = \left(1 - \frac{x}{l}\right)^2$; $\frac{dW}{dx} = -\frac{2}{l}\left(1 - \frac{x}{l}\right)$; $\frac{d^2W}{dx^2} = \frac{2}{l^2}$

$$N = \int_0^l EI \left(\frac{d^2W}{dx^2}\right)^2 dx = \frac{EI_0}{l} \int_0^l x \left(\frac{4}{l^4}\right) dx = \frac{4\,EI_0}{l^3}$$

$$D = \int_0^l \rho A \{W(x)\}^2\, dx = \frac{\rho A_0}{l} \int_0^l x \left(1 - \frac{x}{l}\right)^4 dx = \frac{\rho A_0 l}{30}$$

$$\omega^2 = \frac{N}{D} \simeq \frac{4\,EI_0\,(30)}{l^3(\rho A_0 l)} = \frac{120\, EI_0}{\rho A_0\, l^4}$$

$$\omega \simeq \sqrt{120} \left(\frac{EI_0}{\rho A_0\, l^4}\right)^{1/2}$$

8.40 (a) Equation of motion for a uniform beam is

$$EI \frac{\partial^4 w}{\partial x^4} = f(x,t) - \rho A \frac{\partial^2 w}{\partial t^2} \qquad \text{----}(E_1)$$

Let $\quad w(x,t) = \sum_{n=1}^{\infty} W_n(x)\, T_n(t) \quad$ where $W_n(x) = n^{th}$ normal mode satisfying the boundary conditions and the equation

$$EI \frac{d^4W}{dx^4} = \omega_n^2 \rho A\, W_n \;;\; n = 1, 2, \ldots \qquad \text{----}(E_2)$$

Eq. (E_1) becomes for the assumed solution,

$$EI \sum_{n=1}^{\infty} \frac{d^4W}{dx^4} T_n(t) = f(x,t) - \rho A \sum_{n=1}^{\infty} W_n(x) \cdot \frac{d^2T_n}{dt^2} \qquad \text{----}(E_3)$$

which, in view of (E_2), becomes

$$\sum_{n=1}^{\infty} \omega_n^2\, W_n(x)\, T_n(t) = \frac{1}{\rho A} f(x,t) - \sum_{n=1}^{\infty} W_n(x) \frac{d^2T_n}{dt^2} \qquad \text{----}(E_4)$$

Multiplying Eq. (E_4) by $W_m(x)$ and integrating from 0 to l, we get

$$\frac{d^2T_n}{dt^2} + \omega_n^2 T_n = \frac{2}{\rho A l} F_n(t) \qquad \text{----}(E_5)$$

where

$$F_n(t) = \int_0^l f(x,t)\, W_n(x)\, dx \qquad \text{----}(E_6)$$

Note that the orthogonality of normal modes, namely,

$$\int_0^l W_m(x)\, W_n(x)\, dx = \begin{cases} 0 & \text{if } m = n \\ l/2 & \text{if } m \neq n \end{cases}$$

is used in deriving (E_5). The solution of Eq. (E_5) can be written as

$$T_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{2}{\rho A l \omega_n} \int_0^l F_n(\tau) \sin \omega_n (t-\tau)\, d\tau$$

where A_n and B_n are constants to be determined from initial conditions. ---- (E_7)

Thus the total solution can be expressed as

$$w(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{2}{\rho A l \omega_n} \int_0^t F_n(\tau) \sin \omega_n (t-\tau)\, d\tau \right] W_n(x) \quad \text{---- } (E_8)$$

(b) For a simply supported beam,

$$W_n(x) = \sin \frac{n \pi x}{l}$$

For the given loading, (E_6) becomes $F_n(t) = F_0 \sin \frac{n\pi a}{l} \sin \omega t$

Hence (E_7) becomes

$$T_n(t) = \frac{2 F_0 l^3}{E I \pi^4} \sin \frac{n \pi a}{l} \left[\frac{1}{n^4 - \left(\frac{\omega}{\omega_1}\right)^2} \sin \omega t - \frac{\left(\frac{\omega}{\omega_1}\right)}{n^2 \left\{ n^4 - \left(\frac{\omega}{\omega_1}\right)^2 \right\}} \sin n^2 \omega_1 t \right]$$

where $\omega_1 = \frac{\pi^2}{l^2} \sqrt{\frac{EI}{\rho A}}$ = fundamental frequency of the beam

Thus the forced response of the beam is given by

$$w(x,t) = \frac{2 F_0 l^3}{\pi^4 E I} \left[\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi a}{l}\right)}{n^4 - \left(\frac{\omega}{\omega_1}\right)^2} \sin \frac{n \pi x}{l} \right] \sin \omega t$$

$$- \frac{2 \left(\frac{\omega}{\omega_1}\right) F_0 l^3}{\pi^4 E I} \left[\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi a}{l}\right)}{n^2 \left(n^4 - \left\{\frac{\omega}{\omega_1}\right\}^2 \right)} \sin \frac{n \pi x}{l} \sin n^2 \omega_1 t \right]$$

8.41 Derivation of Eq. (E_5):

If shear deformation is zero, $M = EI \frac{\partial^2 w}{\partial x^2}$

Eq. (8.99) $\Rightarrow$ $$\frac{\partial V}{\partial x} = -\rho A \frac{\partial^2 w}{\partial t^2} + f$$

Eq. (8.100) $\Rightarrow$ $$\frac{\partial M}{\partial x} - V = \rho I \frac{\partial^2 \phi}{\partial t^2}, \quad \frac{\partial^2 M}{\partial x^2} - \frac{\partial V}{\partial x} = \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2}$$

or $$\frac{\partial^2 M}{\partial x^2} + \rho A \frac{\partial^2 w}{\partial t^2} - f = \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2}$$

or $$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} - f = 0 \quad \text{---- (E-1)}$$

Derivation of Eq. (E_6):

Eq. (E.1) becomes, for $f = 0$,

$$\frac{EI}{\rho A}\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} - \frac{I}{A}\frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \qquad \text{---- (E.2)}$$

If $w(x,t) = C \sin \frac{n\pi x}{l} \cos \omega_n t$, Eq. (E.2) becomes

$$\alpha^2 \left(\frac{n\pi}{l}\right)^4 - \omega_n^2 - r^2 \left(\frac{n\pi}{l}\right)^2 \omega_n^2 = 0$$

i.e. $$\omega_n^2 = \frac{\alpha^2 \left(\frac{n\pi}{l}\right)^4}{1 + r^2\left(\frac{n\pi}{l}\right)^2} = \frac{\alpha^2 n^4 \pi^4}{l^4\left(1 + \frac{n^2\pi^2 r^2}{l^2}\right)}$$

$\alpha^2 = \frac{EI}{\rho A}$

$r^2 = \frac{I}{A}$

8.42 Derivation of (E_7):

If rotary inertia is neglected, $$M = EI \frac{\partial \phi}{\partial x} \qquad (E.1)$$

$$V = \kappa AG\left(\phi - \frac{\partial w}{\partial x}\right) \qquad (E.2)$$

Eq. (8.99) $\Rightarrow$ $$-\frac{\partial V}{\partial x} + f = \rho A \frac{\partial^2 w}{\partial t^2} \qquad (E.3)$$

Eq. (8.100) $\Rightarrow$ $$\frac{\partial M}{\partial x} - V = 0 \qquad (E.4)$$

Using (E.1) and (E.2), (E.3) and (E.4) can be written as

$$-\kappa AG\left(\frac{\partial \phi}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) + f = \rho A \frac{\partial^2 w}{\partial t^2}$$

or $$\frac{\partial \phi}{\partial x} = \frac{\partial^2 w}{\partial x^2} + \frac{f}{\kappa AG} - \frac{\rho A}{\kappa AG}\frac{\partial^2 w}{\partial t^2} \qquad (E.5)$$

and $$EI \frac{\partial^2 \phi}{\partial x^2} - \kappa AG\left(\phi - \frac{\partial w}{\partial x}\right) = 0$$

or $$EI \frac{\partial^2}{\partial x^2}\left(\frac{\partial \phi}{\partial x}\right) - \kappa AG \frac{\partial \phi}{\partial x} + \kappa AG \frac{\partial^2 w}{\partial x^2} = 0 \qquad (E.6)$$

Substitution of (E.5) into (E.6) gives

$$EI \frac{\partial^4 w}{\partial x^4} + \frac{EI}{\kappa AG}\frac{\partial^2 f}{\partial x^2} - \frac{\rho A EI}{\kappa AG}\frac{\partial^4 w}{\partial x^2 \partial t^2} - f + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \qquad (E.7)$$

If $f = 0$, (E.7) becomes $$EI\frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \frac{\rho EI}{\kappa G}\frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \qquad (E.8)$$

Derivation of (E_8):

With $w(x,t) = C \sin \frac{n\pi x}{l} \cos \omega_n t$, (E.8) gives

$$\omega_n^2 = \frac{\alpha^2 n^4 \pi^4}{l^4\left(1 + \frac{E}{\kappa G}\cdot\frac{r^2 n^2 \pi^2}{l^2}\right)}$$

8.43 Multiply Eq. (8.83) by $W(x)$ and integrate from 0 to l:

$$\int_0^l W(x) \frac{d^4 W}{dx^4}\, dx = \frac{\omega^2}{c^2}\int_0^l [W(x)]^2\, dx$$

This shows that the sign of ω^2 will be same as that of the left hand side term. Integrating the left hand side expression by parts, we get

$$\frac{d^3W}{dx^3}(x) \cdot W(x) \Big|_0^\ell - \frac{d^2W}{dx^2}(x) \cdot \frac{dW}{dx}(x) \Big|_0^\ell + \int_0^\ell \left[\frac{d^2W}{dx^2}(x) \right]^2 dx \qquad (E_1)$$

For common boundary conditions, the first two terms in (E_1) will be zero, and (E_1) will be positive. Hence ω^2 is positive.

Common boundary conditions:

Simply supported end: $W(x) = \frac{d^2W}{dx^2} = 0$

Fixed end: $W(x) = \frac{dW}{dx} = 0$

Free end: $\frac{d^2W}{dx^2}(x) = \frac{d^3W}{dx^3} = 0$

8.44 Equation of motion:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = F_0 \sin \omega t \qquad (1)$$

For steady state response, we assume the particular solution as

$$w(x,t) = W(x) \sin \omega t \qquad (2)$$

Substitution of Eq. (2) into (1) gives

$$\frac{d^4W}{dx^4} - \frac{\omega^2}{c^2} W = \frac{F_0}{\rho A c^2} \qquad (3)$$

$$\text{where} \quad c^2 = \frac{EI}{\rho A} \qquad (4)$$

The complete solution of Eq. (3) can be written as

$$W(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x - \frac{F_0}{\rho A c^2} \qquad (5)$$

$$\text{where} \quad \beta^2 = \frac{\omega^2}{c^2} \qquad (6)$$

The boundary conditions for a simply supported beam are

$$W(x=0) = 0 \qquad (7)$$

$$\frac{d^2W}{dx^2}(x=0) = 0 \qquad (8)$$

$$W(x=\ell) = 0 \qquad (9)$$

$$\frac{d^2W}{dx^2}(x=\ell) = 0 \qquad (10)$$

Equations (7) and (5) give:

$$C_1 + C_3 - \frac{F_0}{\rho A c^2} = 0 \qquad (11)$$

Equations (8) and (5) yield:

$$- C_1 \beta^2 + C_3 \beta^2 = 0 \quad \text{or} \quad C_1 = C_3 \tag{12}$$

Eqs. (11) and (12) provide:

$$C_1 = C_3 = \frac{F_0}{2 \rho A c^2} \tag{13}$$

Eqs. (9) and (5) give:

$$C_2 \sin \beta \ell + C_4 \sinh \beta \ell = - \frac{F_0}{\rho A c^2} \left(\frac{\cos \beta \ell + \cosh \beta \ell}{2} - 1 \right) \tag{14}$$

Eqs. (10) and (5) yield:

$$- C_2 \sin \beta \ell + C_4 \sinh \beta \ell = \frac{F_0}{2 \rho A c^2} \left(\cos \beta \ell - \cosh \beta \ell \right) \tag{15}$$

Solution of Eqs. (14) and (15) gives:

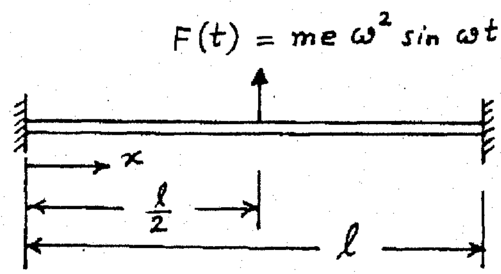
$$C_4 = \frac{F_0}{2 \rho A c^2 \sinh \beta \ell} (1 - \cosh \beta \ell) = - \frac{F_0}{2 \rho A c^2} \tanh \frac{\beta \ell}{2} \tag{16}$$

$$C_2 = \frac{F_0}{2 \rho A c^2 \sin \beta \ell} (1 - \cos \beta \ell) = \frac{F_0}{2 \rho A c^2} \tan \frac{\beta \ell}{2} \tag{17}$$

Thus the complete solution becomes

$$w(x,t) = \frac{F_0}{2 \rho A c^2} \left[\left(\cos \beta x + \cosh \beta x \right) + \tan \frac{\beta \ell}{2} \sin \beta x \right.$$
$$\left. - \tanh \frac{\beta \ell}{2} \sinh \beta x - 2 \right] \sin \omega t$$

8.45



$$\omega = \frac{1000}{60} (2 \pi) = 104.72 \text{ rad/sec.}$$

$$F(t) = m\, e\, \omega^2 \sin \omega t = (0.5)(104.72^2) \sin 104.72\, t = F_0 \sin \omega t \tag{1}$$

where $F_0 = 5483.1392$ N ; $\omega = 104.72$ rad/sec

Steady state response of the beam can be expressed as:

$$w(x,t) = \sum_{n=1}^{\infty} W_n(x)\, q_n(t) \tag{2}$$

where the normal modes, $W_n(x)$, are given by (see Fig. 8.15):

$$W_n(x) = \sinh \beta_n x - \sin \beta_n x + \alpha_n \left(\cosh \beta_n x - \cos \beta_n x \right) \tag{3}$$

$$\text{and} \quad \alpha_n = \frac{\sinh \beta_n \ell - \sin \beta_n \ell}{\cos \beta_n \ell - \cosh \beta_n \ell} \tag{4}$$

The generalized force, $Q_n(t)$, can be expressed as:

$$Q_n(t) = \int_0^\ell f(x,t)\, W_n(x)\, dx = F_0\, W_n(\frac{\ell}{2}) \sin \omega t \tag{5}$$

The steady state values of the generalized coordinates, $q_n(t)$, are given by Eq. (8.117):

$$q_n(t) = \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t-\tau)\, d\tau$$

$$= \frac{F_0 W_n(\frac{\ell}{2})}{\rho A \ell \omega_n} \int_0^t \sin \omega \tau \sin \omega_n (t-\tau)\, d\tau \tag{6}$$

$$\text{where} \quad b = \int_0^\ell W_n^2(x)\, dx = \ell \tag{7}$$

The integral of Eq. (6) can be evaluated as:

$$\int_0^t \sin \omega \tau \left\{ \sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau \right\} d\tau$$

$$= \sin \omega_n t \int_0^t \frac{1}{2} \sin 2 \omega_n \tau\, d\tau - \cos \omega_n t \int_0^t \frac{1}{2} (1 - \cos 2 \omega_n \tau)\, d\tau$$

$$= \frac{\sin \omega_n t}{2 \omega_n} - \frac{t \cos \omega_n t}{2} \tag{8}$$

Thus $q_n(t)$ can be expressed as

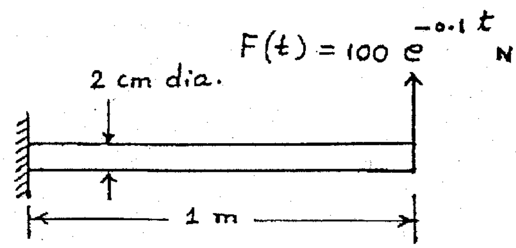
$$q_n(t) = \frac{F_0 W_n(\frac{\ell}{2})}{\rho A \ell \omega_n} \left\{ \frac{\sin \omega_n t}{2 \omega_n} - \frac{t \cos \omega_n t}{2} \right\}$$

$$= \frac{18.2771\, W_n(\frac{\ell}{2})}{\omega_n} \left\{ \sin \omega_n t - t \cos \omega_n t \right\} \tag{9}$$

The steady state response of Eq. (2) can be expressed as

$$w(x,t) = 18.2771 \sum_{n=1}^\infty \frac{W_n(\frac{\ell}{2})}{\omega_n} W_n(x) \left\{ \sin \omega_n t - t \cos \omega_n t \right\} \text{ m} \tag{10}$$

8.46 Steady state displacement of the beam can be expressed as



$$w(x,t) = \sum_{n=1}^\infty W_n(x)\, q_n(t) \tag{1}$$

where the normal modes, $W_n(x)$, are given by (see Fig. 8.15)

$$W_n(x) = C_n \left[\sin \beta_n x - \sinh \beta_n x - \alpha_n (\cos \beta_n x - \cosh \beta_n x)\right] \tag{2}$$

$$\alpha_n = \left(\frac{\sin \beta_n \ell + \sinh \beta_n \ell}{\cos \beta_n \ell + \cosh \beta_n \ell}\right) \tag{3}$$

and the generalized coordinates, $q_n(t)$, are given by

$$\frac{d^2 q_n(t)}{dt^2} + \omega_n^2 q_n(t) = \frac{1}{\rho A b} Q_n(t) \tag{5}$$

$$\text{where} \quad Q_n(t) = \int_0^\ell f(x,t) W_n(x) dx = F(t) W_n(\ell) \tag{6}$$

The steady state solution of Eq. (5) is given by

$$q_n(t) = \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t-\tau) d\tau \tag{7}$$

$$\text{where} \quad b = \int_0^\ell W_n^2(x) dx = \ell \tag{8}$$

$$\int_0^t Q_n(\tau) \sin \omega_n (t-\tau) d\tau = 100 W_n(\ell) \int_0^t e^{-0.1\tau} \sin \omega_n (t-\tau) d\tau$$

$$= -100 W_n(\ell) e^{-0.1t} \int_{t-\tau=t}^{t-\tau=0} e^{0.1(t-\tau)} \sin \omega_n (t-\tau) (-d\tau) \tag{9}$$

Using the formula

$$\int e^{ax} \sin bx\, dx = \frac{1}{a^2+b^2} e^{ax} \left\{a \sin bx - b \cos bx\right\} \tag{10}$$

Eq. (9) can be evaluated to obtain

$$q_n(t) = \frac{100 W_n(\ell)}{\rho A \ell \omega_n (\omega_n^2 + 0{,}01)} \left\{\omega_n e^{-0.1t} + 0.1 \sin \omega_n t - \omega_n \cos \omega_n t\right\} \tag{11}$$

Using $\rho = 7500$ kg/m^3, $\ell = 1$ m, $A = \frac{\pi}{4}(0.02^2) = 3.1416\,(10^{-4})$ m^2, Eq. (11) can be written as

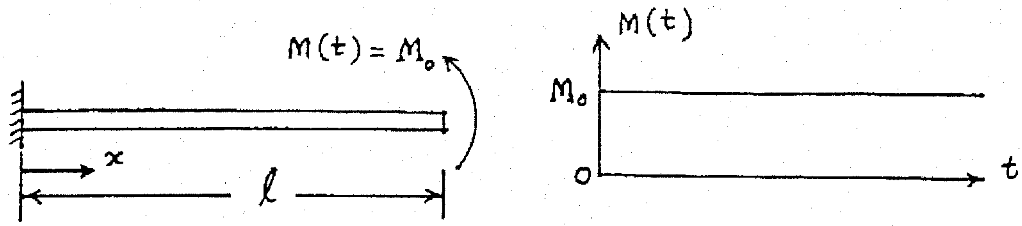
$$q_n(t) = \frac{42.4412 W_n(\ell)}{\omega_n (\omega_n^2 + 0.01)} \left\{\omega_n e^{-0.1t} + 0.1 \sin \omega_n t - \omega_n \cos \omega_n t\right\} \tag{12}$$

The total steady state solution can be found from Eq. (1).

8.47 The solution is assumed as

$$w(x,t) = \sum_{n-1}^{\infty} W_n(x) q_n(t) \tag{1}$$

where the normal modes of a cantilever beam are given by (Fig. 8.15):



$$W_n(x) = \sin\beta_n x - \sinh\beta_n x - \alpha_n(\cos\beta_n x - \cosh\beta_n x) \tag{2}$$

$$\alpha_n = \left(\frac{\sin\beta_n \ell + \sinh\beta_n \ell}{\cos\beta_n \ell + \cosh\beta_n \ell}\right) \tag{3}$$

where $\beta_n \ell$ are given by the frequency equation:

$$\cos\beta_n \ell \cosh\beta_n \ell = -1 \tag{4}$$

The generalized force $Q_n(t)$ given by Eq. (8.115) becomes

$$Q_n(t) = M_0 \frac{dW_n}{dx}\Big|_{x=\ell} \tag{5}$$

where

$$\frac{dW_n}{dx}\Big|_{x=\ell} = \beta_n(\cos\beta_n \ell - \cosh\beta_n \ell) + \alpha_n \beta_n (\sin\beta_n \ell + \sinh\beta_n \ell) \tag{6}$$

The steady state response of the beam is given by Eq. (1) with

$$\begin{aligned} q_n(t) &= \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin\omega_n(t-\tau)\, d\tau \\ &= \frac{1}{\rho A b \omega_n} M_0 \frac{dW_n}{dx}\Big|_{x=\ell} \int_0^t \sin\omega_n(t-\tau)\, d\tau \end{aligned} \tag{7}$$

$$\text{where} \quad b = \int_0^\ell W_n^2(x)\, dx = \ell \tag{8}$$

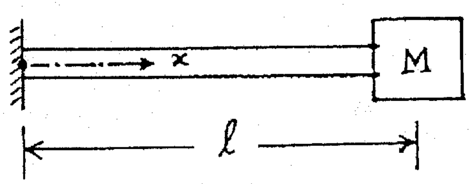
Noting that

$$\int_0^t \sin\omega_n(t-\tau)\, d\tau = \frac{1}{\omega_n}(1 - \cos\omega_n t) \tag{9}$$

Eq. (7) can be written as

$$q_n(t) = \frac{M_0}{\rho A \ell \omega_n^2} \frac{dW_n}{dx}\Big|_{x=\ell} (1 - \cos\omega_n t) \tag{10}$$

8.48



$$w(x,t) = W(x) \sin\omega t \tag{1}$$

where $\quad W(x) = C_1 \cos\beta x + C_2 \sin\beta x + C_3 \cosh\beta x + C_4 \sinh\beta x \qquad (2)$

Boundary conditions:

$$w(0,\ t) = 0 \tag{3}$$

$$\frac{\partial w}{\partial x}(0,\ t) = 0 \tag{4}$$

$$\frac{\partial^2 w}{\partial x^2}(x = \ell,\ t) = 0 \tag{5}$$

$$EI\ \frac{\partial^3 w}{\partial x^3}(x = \ell,\ t) = m\ \frac{\partial^2 w}{\partial t^2}(x = \ell,\ t) \tag{6}$$

Eq. (2) gives:

$$\frac{dW}{dx} = -\beta\, C_1 \sin\beta x + \beta\, C_2 \cos\beta x + \beta\, C_3 \sinh\beta x + \beta\, C_4 \cosh\beta x \tag{7}$$

$$\frac{d^2W}{dx^2} = \beta^2 \left\{- C_1 \cos\beta x - C_2 \sin\beta x + C_3 \cosh\beta x + C_4 \sinh\beta x\right\} \tag{8}$$

$$\frac{d^3W}{dx^3} = \beta^3 \left\{C_1 \sin\beta x - C_2 \cos\beta x + C_3 \sinh\beta x + C_4 \cosh\beta x\right\} \tag{9}$$

Eqs. (2) and (3) give:

$$C_1 + C_3 = 0 \tag{10}$$

Eqs. (2) and (4) lead to:

$$C_2 + C_4 = 0 \tag{11}$$

Eqs. (2) and (5) yield:

$$- C_1 \cos\beta\ell - C_2 \sin\beta\ell + C_3 \cosh\beta\ell + C_4 \sinh\beta\ell = 0 \tag{12}$$

Eqs. (2) and (6) result in:

$$\begin{gathered} C_1 (\sin\beta\ell + k\cos\beta\ell) + C_2 (-\cos\beta\ell + k\sin\beta\ell) \\ + C_3 (\sinh\beta\ell + k\cosh\beta\ell) + C_4 (\cosh\beta\ell + k\sinh\beta\ell) = 0 \end{gathered} \tag{13}$$

where

$$k = \frac{m\,\omega^2}{E\,I\,\beta^3} \tag{14}$$

Eqs. (10) to (13) can be expressed in matrix form as

$$[A]\,\vec{C} = \vec{0} \tag{15}$$

where

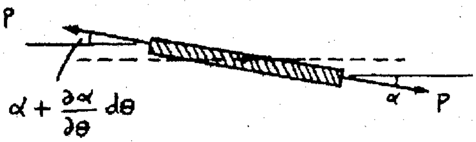
$$[A] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\cos\beta\ell & -\sin\beta\ell & \cosh\beta\ell & \sinh\beta\ell \\ (\sin\beta\ell & (-\cos\beta\ell & (\sinh\beta\ell & (\cosh\beta\ell \\ +k\cos\beta\ell) & +k\sin\beta\ell) & +k\cosh\beta\ell) & +k\sinh\beta\ell) \end{bmatrix} \tag{16}$$

$$\vec{C} = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} \quad ; \quad \vec{0} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

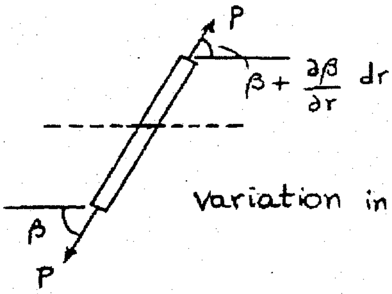
By setting the determinant of the coefficient matrix, [A], to zero, we obtain the frequency equation:

$$\Big|[A]\Big| = 0 \tag{17}$$

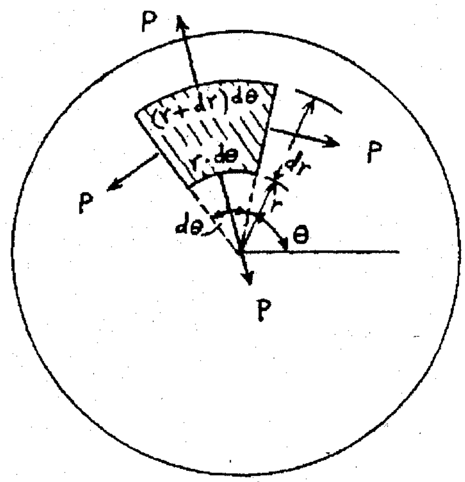
8.49



variation in θ-direction



variation in r-direction



Let P = tension

Forces in radial direction are $\quad P\, r\, d\theta$ and $P(r+dr)\, d\theta$

vertical component of radial forces

$$= P(r+dr)\,d\theta\left(\beta + \frac{\partial \beta}{\partial r} dr\right) - P\, r\, d\theta\, \beta = P\, r\left(\frac{\partial \beta}{\partial r} + \frac{1}{r}\beta\right) dr\, d\theta$$

$$= P\, r\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) dr\, d\theta \qquad \text{since } \beta = \frac{\partial w}{\partial r}$$

Forces in tangential direction are $\quad P\,dr$ and $P\,dr$.

vertical component of tangential forces

$$= P\, dr\left(\alpha + \frac{\partial \alpha}{\partial \theta} d\theta\right) - P\, dr\, \alpha = P\,\frac{\partial \alpha}{\partial \theta}\, dr\, d\theta$$

$$= P \cdot \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\, r\, dr\, d\theta \qquad \text{since } \alpha = \frac{1}{r}\frac{\partial w}{\partial \theta} \text{ and } \frac{\partial \alpha}{\partial \theta} = \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}$$

Equating the total vertical force to mass times acceleration, we get

$$P\, r\, dr\, d\theta\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right] = \rho\,\frac{\partial^2 w}{\partial t^2}\, r\, dr\, d\theta$$

i.e.

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{\rho}{P}\,\frac{\partial^2 w}{\partial t^2}$$

8.50 For harmonic motion, equation of motion becomes

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} = -\frac{\rho}{P}\omega^2 W$$

Let $W(r,\theta) = X(r)\, Y(\theta)$

$$\frac{r^2}{X}\left(\frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} + \frac{\rho\omega^2}{P} X\right) = -\frac{1}{Y}\frac{d^2Y}{d\theta^2} = \alpha^2 \text{ (say)}$$

$$\therefore \quad \frac{d^2Y}{d\theta^2} + \alpha^2 Y = 0 \qquad (E_1)$$

$$\frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} + \left(\frac{\rho\omega^2}{P} - \frac{\alpha^2}{r^2}\right) X = 0 \qquad (E_2)$$

Solution of (E_2) is $\quad X(r) = C_1\, J_m(\gamma \cdot r) + C_2\, I_m(\gamma\cdot r)\,;\ m = 0,1,2,\ldots$

where J_m and I_m are Bessel functions of the first and second kinds, respectively, and $\quad \gamma = \frac{\rho\,\omega^2}{P}$

Since $I_m(\gamma\cdot r) \to \infty$ when $r \to 0$, $C_2 = 0$ to keep $X(r)$ and hence $w(r,\theta,t)$ finite.

$$\therefore\ X(r) = C_1\, J_m(\gamma\cdot r)$$

At $r = R$, $X(r) = 0$ i.e., $\quad J_m(\gamma\cdot R) = 0 \qquad (E_3)$

Eq. (E_3) has several roots: $\gamma_1 R,\ \gamma_2 R, \ldots, \gamma_n R, \ldots$

$$\therefore\ \omega_{mn}^2 = \frac{\gamma_n\, P}{\rho}$$

8.51 (a) Equation of motion: $\quad \rho\,\frac{\partial^2 w}{\partial t^2} = P\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + f \qquad (E_1)$

Let the forced response of the rectangular membrane be of the form

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{mn}(t)\, \sin\frac{m\pi x}{a}\, \sin\frac{n\pi y}{b} \qquad (E_2)$$

where $T_{mn}(t)$ is to be determined. Eqs. (E_1) and (E_2) give

$$\pi^2 P \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) T_{mn}(t)\,\sin\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}$$

$$+\ \rho \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{d^2 T_{mn}(t)}{dt^2}\,\sin\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b} - f = 0 \qquad (E_3)$$

Multiply (E_3) by $\sin\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}$ and integrate with respect to x from 0 to a and with respect to y from 0 to y to get

$$\frac{d^2 T_{mn}(t)}{dt^2} + \frac{P\pi^2}{\rho}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) T_{mn}(t) = \frac{4}{ab\rho}\, F_{mn}(t)\,;$$

$$m, n = 1, 2, \ldots \qquad (E_4)$$

where
$$F_{mn}(t) = \int_0^a \int_0^b f(x,y,t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy \qquad (E_5)$$

Solution of (E_4) can be expressed as

$$T_{mn}(t) = A_{mn} \cos \pi \sqrt{\frac{P}{\rho}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\, t + B_{mn} \sin \pi \sqrt{\frac{P}{\rho}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\, t$$
$$+ \frac{4}{\pi \rho a b \sqrt{\frac{P}{\rho}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}} \int_0^t F_{mn}(\tau) . \sin\left\{\pi \sqrt{\frac{P}{\rho}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\,(t-\tau)\right\} d\tau \qquad (E_6)$$

where A_{mn} and B_{mn} are determined from the known initial conditions of the membrane. Thus the general solution is given by Eq. (E_2) with $T_{mn}(t)$ shown in the last equation, (E_6).

(b) For $f(x,y,t) = f_0$, (E_5) becomes

$$F_{mn}(t) = \int_0^a \int_0^b f_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy$$
$$= \frac{f_0 \, ab}{\pi^2 mn} (1-\cos m\pi)(1-\cos n\pi)$$
$$= \begin{cases} 0 & \text{for } m \text{ or } n \text{ even} \\ \dfrac{4 f_0 ab}{\pi^2 mn} & \text{for } m \text{ and } n \text{ odd} \end{cases}$$

Since the membrane is at rest initially, $A_{mn} = B_{mn} = 0$ and Eq. (E_6) gives

$$T_{mn}(t) = \frac{4}{\pi \rho a b \Delta} * \frac{4 f_0 ab}{\pi^2 mn} \int_0^t \sin \pi \Delta . (t-\tau) . d\tau$$
$$= \frac{16 f_0}{\pi^3 mn \rho \Delta} \times \frac{(1-\cos \pi \Delta t)}{\pi \Delta}$$
$$= \frac{16 f_0}{\pi^4 mn \rho \Delta^2} (1-\cos \pi \Delta t) \qquad \text{for } m \text{ and } n \text{ odd}$$

where $\Delta = \left\{\frac{P}{\rho}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\right\}^{1/2}$

Final solution is

$$w(x,y,t) = \frac{16 f_0}{\pi^4 P} \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} \frac{1}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} . (1-\cos \pi \Delta t)$$

8.52 From example 8.11,

$$w(x,y,t) = X(x)\, Y(y)\, T(t) = (C_1 \cos \alpha x + C_2 \sin \alpha x)(C_3 \cos \beta y + C_4 \sin \beta y) * (A \cos \omega t + B \sin \omega t)$$

Boundary conditions:

$$w(0,y,t) = w(a,y,t) = 0\,; \quad 0 \le y \le b \text{ and } t \ge 0$$

$$w(x,0,t) = w(x,b,t) = 0\,; \quad 0 \le x \le a \text{ and } t \ge 0$$

i.e. $X(0) = X(a) = 0\,, \quad Y(0) = Y(b) = 0$

i.e. $C_1 = 0,\ C_3 = 0 \Rightarrow \sin\alpha a = 0,\ \sin\beta b = 0$

$$\alpha_m = \frac{m\pi}{a}\ (m = 1,2,\ldots),\quad \beta_n = \frac{n\pi}{b}\ (n = 1,2,\ldots)$$

$$\therefore\ \omega_{mn}^2 = c^2\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\ ; \quad m, n = 1, 2, \ldots$$

and the corresponding displacement solution is

$$w_{mn}(x,y,t) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left(A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t\right)$$

General solution is

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}(x,y,t)$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left(A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t\right)$$

8.53 General solution

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left(A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t\right) \tag{E$_1$}$$

If $w(x,y,0) = w_0(x,y)$ and $\frac{\partial w}{\partial t}(x,y,0) = \dot{w}_0(x,y)$ $\Big\}$; $0 \le x \le a,\ 0 \le y \le b$ (E$_2$)

Eq. (E$_1$) gives

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = w_0(x,y) \tag{E$_3$}$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\omega_{mn}B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = \dot{w}_0(x,y) \tag{E$_4$}$$

These are double Fourier sine series expansions so that

$$A_{mn} = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b} w_0(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

$$= \frac{4}{ab}\int_{x=0}^{a} w_0\sin\frac{\pi x}{a}\sin\frac{m\pi x}{a}\,dx\int_{y=0}^{b}\sin\frac{\pi y}{b}\sin\frac{n\pi y}{b}\,dy \tag{E$_5$}$$

$$B_{mn} = \frac{4}{ab\,\omega_{mn}}\int_{x=0}^{a}\int_{y=0}^{b}\dot{w}_0(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

$$= 0 \text{ for given data} \tag{E$_6$}$$

Using the relation $\int \sin \alpha t \, \sin \beta t \, dt = \frac{1}{2}\left\{ \frac{\sin(\alpha-\beta)t}{\alpha-\beta} - \frac{\sin(\alpha+\beta)t}{\alpha+\beta} \right\}; \; \alpha \neq \beta$

Eq. (E_5) can be simplified as

$$A_{mn} = \frac{4 w_0}{ab} \left\{ \frac{1}{2} \left[\frac{\sin \frac{(m-1)\pi x}{a}}{\frac{(m-1)\pi}{a}} - \frac{\sin \frac{(m+1)\pi x}{a}}{\frac{(m+1)\pi}{a}} \right]_0^a \right\} \left\{ \frac{1}{2} \left[\frac{\sin \frac{(n-1)\pi y}{b}}{\frac{(n-1)\pi}{b}} - \frac{\sin \frac{(n+1)\pi y}{b}}{\frac{(n+1)\pi}{b}} \right]_0^b \right\}$$

$$= 0 \quad \text{for } m > 1 \text{ and/or } n > 1 \qquad (E_7)$$

For $m = 1$ and $n = 1$, Eq. (E_5) can be simplified as

$$A_{11} = \frac{4 w_0}{ab} \int_0^a \sin^2 \frac{\pi x}{a} \, dx \int_0^b \sin^2 \frac{\pi y}{b} \, dy = w_0 \qquad (E_8)$$

$$\therefore \quad w(x,y,t) = w_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \cos \omega_{11} t$$

with $\omega_{11} = \left\{ c^2 \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \right\}^{1/2}$

(from the solution of problem 8.52)

8.54 General solution:

$$w(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left(A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t \right)$$

If $w(x,y,0) = w_0(x,y)$
and $\frac{\partial w}{\partial t}(x,y,0) = \dot{w}_0(x,y)$ $\Big\}$; $0 \le x \le a, \quad 0 \le y \le b$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = w_0(x,y)$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{mn} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \dot{w}_0(x,y)$$

These are double Fourier sine series expansions of $w_0(x,y)$ and $\dot{w}_0(x,y)$.

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b w_0(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy$$

$$= 0 \text{ for given data}$$

$$B_{mn} = \frac{4}{ab\,\omega_{mn}} \int_0^a \int_0^b \dot{w}_0(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy$$

$$= \frac{4 \dot{w}_0}{ab\,\omega_{mn}} \int_0^a \sin \frac{m\pi x}{a} \sin \frac{\pi x}{a} \, dx \int_0^b \sin \frac{n\pi y}{b} \sin \frac{2\pi y}{b} \, dy \quad \text{---} (E_1)$$

using the relation

$$\int \sin \alpha t \cdot \sin \beta t \cdot dt = \frac{1}{2} \left\{ \frac{\sin(\alpha-\beta)t}{\alpha-\beta} - \frac{\sin(\alpha+\beta)t}{\alpha+\beta} \right\} \text{ for } \alpha \neq \beta,$$

Eq. (E_1) becomes

$$B_{mn} = \frac{4\dot{w}_0}{ab\,\omega_{mn}} \left\{ \frac{1}{2} \left(\frac{\sin\frac{(m-1)\pi x}{a}}{m-1} - \frac{\sin\frac{(m+1)\pi x}{a}}{m+1} \right)\Bigg|_0^a \right\} \times$$

$$\left\{ \frac{1}{2} \left(\frac{\sin\frac{(n-2)\pi y}{b}}{n-2} - \frac{\sin\frac{(n+2)\pi y}{b}}{n+2} \right)\Bigg|_0^b \right\}$$

$$= 0 \quad \text{for } m \neq 1 \text{ and } n \neq 2.$$

$$B_{12} = \frac{4\dot{w}_0}{ab\,\omega_{mn}} \left(\int_0^a \sin^2\frac{\pi x}{a}\,dx \right)\left(\int_0^b \sin^2\frac{2\pi y}{b}\,dy \right)$$

$$= \frac{4\dot{w}_0}{ab\,\omega_{mn}} \left\{ \left(\frac{x}{2} - \frac{\sin\frac{\pi x}{a}\cos\frac{\pi x}{a}}{\left(\frac{2\pi}{a}\right)} \right)_0^a \right\} \left\{ \left(\frac{y}{2} - \frac{\sin\frac{2\pi y}{b}\cos\frac{2\pi y}{b}}{\left(\frac{4\pi}{b}\right)} \right)_0^b \right\}$$

$$= \frac{4\dot{w}_0}{ab\,\omega_{12}} \left(\frac{a}{2}\right)\left(\frac{b}{2}\right) = \frac{\dot{w}_0}{\omega_{12}}$$

$$\therefore \quad w(x,y,t) = \frac{\dot{w}_0}{\omega_{12}} \sin\frac{\pi x}{a} \sin\frac{2\pi y}{b} \sin\omega_{12} t$$

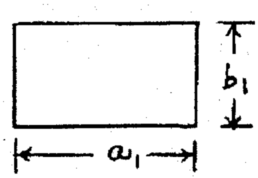
8.55 Fundamental natural frequency of transverse vibration is given by :

$$\omega_{11}^2 = c^2\pi^2\left(\frac{1}{a_1^2} + \frac{1}{b_1^2}\right) \quad \text{for rectangular membrane of sides } a_1 \text{ and } b_1 \qquad \cdots (E_1)$$

with $\quad c^2 = P/\rho \qquad \cdots (E_2)$

(a) For $a_1 = b_1 = a$, $\quad \omega_{11}^2 = c^2\pi^2\left(\frac{2}{a^2}\right) \qquad \cdots (E_3)$

(c) For $a_1 = 2b_1$, $\quad$ area $= a_1 b_1 = a^2 = 2b_1^2$



$\therefore\; b_1 = a/\sqrt{2}$, $\quad a_1 = \sqrt{2}\,a$

$$\omega_{11}^2 = c^2\pi^2\left(\frac{1}{a_1^2} + \frac{1}{b_1^2}\right) = c^2\pi^2\left(\frac{5}{2a^2}\right) \qquad \cdots (E_4)$$

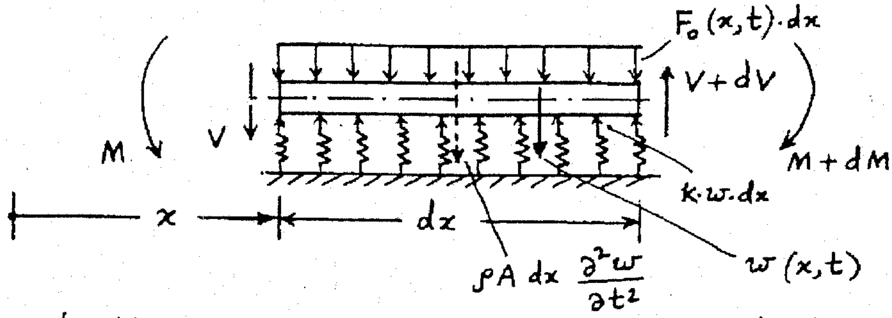
(b) From solution of problem 8.50, the fundamental natural frequency of a circular membrane of radius R is given by

$$\omega_{01}^2 = \frac{\gamma_1 P}{\rho} = c^2\gamma_1 \quad \text{where } \gamma_1 R = 2.404 = \text{first zero of Bessel function of the first kind.}$$

Since $\pi R^2 = a^2$, $R = a/\sqrt{\pi}$ and

$$\omega_{01}^2 = c^2\left(\frac{2.404}{R}\right) = c^2\left(\frac{4.2610}{a}\right) \qquad \cdots (E_5)$$

8.56



(a) If load due to the car moves with a constant velocity v_o in the x-direction,

$$F(x,t) = F(x - v_o t) \qquad (E_1)$$

Equilibrium equations are

$$(V+dV) - V - F_o(x,t)\, dx + k\, w\, dx = -\rho A\, dx \frac{\partial^2 w}{\partial t^2}$$

or

$$\frac{\partial V}{\partial x} - F_o(x,t) + k\, w = -\rho A \frac{\partial^2 w}{\partial t^2} \qquad (E_2)$$

and

$$(M+dM) - M - (V+dV)\, dx + F(x,t)\, dx \cdot \frac{dx}{2} = 0$$

or

$$\frac{\partial M}{\partial x} - V = 0 \qquad (E_3)$$

Using (E_3), (E_2) can be rewritten as

$$\frac{\partial^2 M}{\partial x^2} - F_o(x,t) + k\, w = -\rho A \frac{\partial^2 w}{\partial t^2} \qquad (E_4)$$

But

$$M = EI \frac{\partial^2 w}{\partial x^2} \qquad (E_5)$$

Eqs. (E_4) and (E_5) lead to

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) - F_o(x,t) + k\, w = -\rho A \frac{\partial^2 w}{\partial t^2} \qquad (E_6)$$

For a uniform beam, (E_6) simplifies to, in view of (E_1),

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} + k\, w = F_o(x - v_o t) \qquad (E_7)$$

(b) Solution:

Defining $y = x - v_o t \qquad (E_8)$

Eq. (E_7) can be expressed as

$$EI \frac{d^4 w}{dy^4} + \rho A v_o^2 \frac{d^2 w}{dy^2} + k\, w = F_o(y) \qquad (E_9)$$

Let F_o = concentrated load. Then the governing equation at all points of the beam, except at $y = 0$, is

$$EI \frac{d^4 w}{dy^4} + \rho A v_o^2 \frac{d^2 w}{dy^2} + k\, w = 0 \qquad (E_{10})$$

Solution of (E_{10}) can be expressed as

$$w(y) = e^{\alpha y} \tag{E$_{11}$}$$

Substitution of (E_{11}) into (E_{10}) gives

$$EI\,\alpha^4 + \rho A v_0^2 + k = 0 \tag{E$_{12}$}$$

whose roots can be expressed as

$$\alpha_{1,2} = \pm (a + i b) \quad , \quad \alpha_{3,4} = \pm (a - i b) \tag{E$_{13}$}$$

with

$$\left.\begin{array}{l} a = \sqrt{1-c}\; d \;, \quad b = \sqrt{1+c}\; d \;, \quad c = \dfrac{v_0^2}{\sqrt{\dfrac{4EIk}{\rho^2 A^2}}} \\ \text{and} \quad d = \left(\dfrac{k}{4EI}\right)^{1/4} \end{array}\right\} \tag{E$_{14}$}$$

Thus the solution of (E_{10}) becomes

$$w(y) = A_1 e^{\alpha_1 y} + A_2 e^{\alpha_2 y} + A_3 e^{\alpha_3 y} + A_4 e^{\alpha_4 y} \tag{E$_{15}$}$$

where A_i $(i = 1, 2, 3, 4)$ are constants which can be determined from the following conditions:

$$\left.\begin{array}{l} w = 0 \quad \text{at} \quad y = \infty \\ \dfrac{d^2 w}{dy^2} = 0 \quad \text{at} \quad y = \infty \end{array}\right\} \text{deflection \& bending moment are zero at } y = \infty$$

$$\left.\dfrac{dw}{dy} = 0 \quad \text{at} \quad y = 0 \right\} \text{slope is zero under the load}$$

$$\left.\begin{array}{l} EI \left.\dfrac{d^3 w}{dy^3}\right|_{y=0^+} - EI \left.\dfrac{d^3 w}{dy^3}\right|_{y=0^-} = P \\ \text{i.e.,} \quad EI \dfrac{d^2 w}{dy^2} = \dfrac{P}{2} \end{array}\right\} \text{shear force has discontinuity under the load}$$

8.57 $W(x) = \dfrac{c_0 x^2}{24 EI} (l - x)^2 = \dfrac{c_0}{24 EI} (x^2 l^2 + x^4 - 2 l x^3)$

$$\frac{dW}{dx} = \frac{c_0}{24 EI} (2 x l^2 + 4 x^3 - 6 l x^2)$$

$$\frac{d^2 W}{dx^2} = \frac{c_0}{24 EI} (2 l^2 + 12 x^2 - 12 l x)$$

$$N = EI \int_0^l \left(\frac{d^2 W}{dx^2}\right)^2 dx = 4 EI \left(\frac{c_0}{24 EI}\right)^2 \int_0^l (l^2 + 6 x^2 - 6 l x)^2 \, dx$$

$$= \frac{c_0^2 \, l^5}{720 \, EI} \tag{E$_1$}$$

$$D = \rho A \int_0^l (W(x))^2 dx = \rho A \left(\frac{c_0}{24 EI}\right)^2 \int_0^l (x^2 l^2 + x^4 - 2 l x^3)^2 dx$$

$$= \frac{\rho A \, l^9 \, c_0^2}{362\,880 \; E^2 I^2} \tag{E$_2$}$$

$$\omega^2 = \frac{N}{D} = \left(\frac{c_0^2 \ell^5}{720 EI}\right)\left(\frac{362880\ E^2 I^2}{\rho A \ell^9 c_0^2}\right) = 504 \frac{EI}{\rho A \ell^4}$$

$$\therefore \quad \omega = 22.4499 \sqrt{\frac{EI}{\rho A \ell^4}}$$

This can be compared with the exact solution (Fig. 8.15):

$$\omega_{exact} = (4.730041)^2 \sqrt{\frac{EI}{\rho A \ell^4}} = 22.3733 \sqrt{\frac{EI}{\rho A \ell^4}} .$$

8.58 $W(x) = c_0 \left(1 - \cos \frac{2\pi x}{\ell}\right)$

$$\frac{dW}{dx} = c_0 \frac{2\pi}{\ell} \sin \frac{2\pi x}{\ell} \quad ; \quad \frac{d^2W}{dx^2} = c_0 \left(\frac{2\pi}{\ell}\right)^2 \cos \frac{2\pi x}{\ell}$$

$$N = EI \int_0^\ell \left(\frac{d^2W}{dx^2}\right)^2 dx = EI\ c_0^2 \left(\frac{2\pi}{\ell}\right)^4 \int_0^\ell \cos^2 \frac{2\pi x}{\ell} . dx$$

$$= 8 EI\ c_0^2 \pi^4 / \ell^3$$

$$D = \rho A \int_0^\ell (W(x))^2 dx = \rho A\ c_0^2 \int_0^\ell \left(1 + \cos^2 \frac{2\pi x}{\ell} - 2\cos \frac{2\pi x}{\ell}\right) dx$$

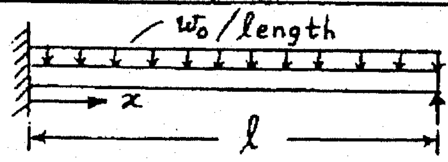
$$= 3 c_0^2 \rho A \ell / 2$$

$$\omega^2 = \frac{N}{D} = \frac{16\pi^4}{3} \cdot \frac{EI}{\rho A \ell^4}$$

$\therefore \omega = 22.7930 \sqrt{\frac{EI}{\rho A \ell^4}}$. Compare this with

$$\omega_{exact} = (4.730041)^2 \sqrt{\frac{EI}{\rho A \ell^4}} = 22.3733 \sqrt{\frac{EI}{\rho A \ell^4}} .$$

8.59 From strength of materials, the static deflection curve is given by



$$W(x) = \frac{w_0 x^2}{48EI} (\ell - x)(2x - 3\ell) = \frac{c_0}{48EI} (5\ell x^3 - 2x^4 - 3\ell^2 x^2)$$

$$\frac{dW}{dx} = \frac{c_0}{48EI} (15\ell x^2 - 8x^3 - 6\ell^2 x)$$

$$\frac{d^2W}{dx^2} = \frac{c_0}{8EI} (5\ell x - 4x^2 - \ell^2)$$

$$N = EI \int_0^\ell \left(\frac{d^2W}{dx^2}\right)^2 dx = \frac{c_0^2 \ell^5}{320\ EI}$$

$$D = \rho A \int_0^l (W(x))^2 dx = 1.3090 \times 10^{-5} \frac{c_0^2 \rho A l^9}{E^2 I^2}$$

$$\omega^2 = \frac{N}{D} = 238.7319 \frac{EI}{\rho A l^4}$$

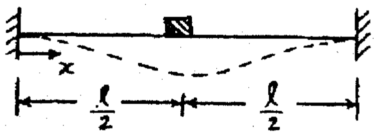
$$\therefore \quad \omega = 15.4510 \sqrt{\frac{EI}{\rho A l^4}} .$$

This can be compared with the exact value

$$\omega_{exact} = (3.926602)^2 \sqrt{\frac{EI}{\rho A l^4}} = 15.4182 \sqrt{\frac{EI}{\rho A l^4}} .$$

8.60

$$W(x) = \begin{cases} \frac{x^2}{48EI}(-4x+3l) \; ; & 0 \le x \le \frac{l}{2} \\ \frac{(l-x)^2}{48EI}(4x-l) \; ; & \frac{l}{2} \le x \le l \end{cases}$$



$$\frac{d^2 W}{dx^2} = \begin{cases} \frac{1}{8EI}(-4x+l) \; ; & 0 \le x \le \frac{l}{2} \\ \frac{1}{8EI}(4x-3l) \; ; & \frac{l}{2} \le x \le l \end{cases}$$

$$T_{max} \text{ of beam } = \frac{\omega^2}{2} \int_0^l \rho A \, [W(x)]^2 \, dx$$

$$= \frac{\omega^2 \rho A}{2} \left(\frac{1}{48EI}\right)^2 \left[\int_0^{l/2} x^4 (16x^2 + 9l^2 - 24xl) dx + \int_{l/2}^{l} (l-x)^4 (4x-l)^2 dx \right]$$

$$= \frac{\rho A \omega^2}{4608 E^2 I^2} \left\{ \left[\frac{16x^7}{7} + \frac{9l^2x^5}{5} - \frac{24 l x^6}{6} \right]_0^{l/2} + \left[\frac{16x^7}{7} - \frac{72 l x^6}{6} + \frac{129 l^2 x^5}{5} - \frac{116 l^3 x^4}{4} + \frac{54 l^4 x^3}{3} - \frac{12 l^5 x^2}{2} + l^6 x \right]_{l/2}^{l} \right\}$$

$$= 5 \times 10^{-6} \left(\frac{\rho A \omega^2 l^7}{E^2 I^2} \right)$$

$$T_{max} \text{ of mass } = \frac{1}{2} m \, \dot{w}_{max}\Big|_{x=\frac{l}{2}} = \frac{1}{2} m \omega^2 W^2(x = \tfrac{l}{2})$$

$$= \frac{1}{2} m \omega^2 \left(\frac{l^3}{192 EI} \right)^2 = 13.6 \times 10^{-6} \left(\frac{m \omega^2 l^6}{E^2 I^2} \right)$$

$$T_{max} \text{ total } = 5 \times 10^{-6} \frac{\rho A \omega^2 l^7}{E^2 I^2} + 13.6 \times 10^{-6} \frac{m \omega^2 l^6}{E^2 I^2}$$

$$V_{max} = \frac{1}{2}\int_0^l EI\left(\frac{d^2W}{dx^2}\right)^2 dx$$

$$= \frac{1}{128\,EI}\left[\int_0^{\frac{l}{2}} (16x^2 + l^2 - 8lx)\,dx + \int_{l/2}^{l} (16x^2 + 9l^2 - 24lx)\,dx\right]$$

$$= \frac{1}{128\,EI}\left[\left(\frac{16}{3}x^3 + l^2 x - \frac{8lx^2}{2}\right)_0^{l/2} + \left(\frac{16}{3}x^3 + 9l^2 x - \frac{24lx^2}{2}\right)_{l/2}^{l}\right]$$

$$= \frac{l^3}{384\,EI}$$

T_{max} total $= V_{max}$ gives

$$\omega^2 = \frac{l^3}{384\,EI}\cdot\frac{E^2 I^2}{\left(5\times10^{-6}\rho A l^7 + 13.6\times10^{-6}\, m\, l^6\right)} = \frac{2590.7\ EI}{l^3(5\rho A l + 13.6\, m)}$$

$$\therefore\ \omega = 50.8987\sqrt{\frac{EI}{l^3(5M_b + 13.6\, m)}}$$ where $M_b = \rho A l$ = mass of beam

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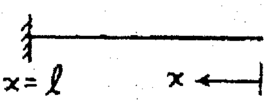
8.61 Let $W(x) = c_0\left(1 - \frac{x}{l}\right)^2 = c_0\left(1 + \frac{x^2}{l^2} - \frac{2x}{l}\right)$



$$\frac{d^2W}{dx^2} = 2c_0\left(\frac{1}{l^2}\right)$$

$$T_{max} = \frac{\omega^2}{2}\int_0^l \rho A(x)\,[W(x)]^2 dx = \frac{\rho\omega^2 A_0 c_0^2}{2l}\int_0^l x\left(1 + \frac{x^2}{l^2} - \frac{2x}{l}\right)^2 dx$$

$$= \frac{\rho\omega^2 A_0 c_0^2}{2l}\left[\frac{x^2}{2} + \frac{x^6}{6l^4} + \frac{4x^4}{4l^2} + \frac{2x^4}{4l^2} - \frac{4x^5}{5l^3} - \frac{4x^3}{3l}\right]_0^l$$

$$= \frac{\rho\omega^2 A_0 c_0^2\, l}{60}$$

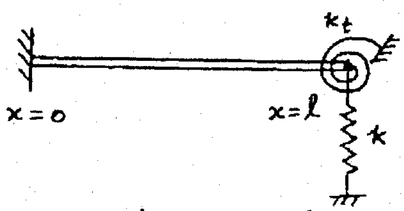
$$V_{max} = \frac{1}{2}\int_0^l EI(x)\left(\frac{d^2W}{dx^2}\right)^2 dx = \frac{E}{2}\int_0^l\left(\frac{I_0 x}{l}\right)\left(\frac{4c_0^2}{l^4}\right)dx = \frac{EI_0 c_0^2}{l^3}$$

$T_{max} = V_{max}$ gives

$$\omega^2 = \frac{EI_0 c_0^2}{l^3}\cdot\frac{60}{\rho A_0 c_0^2 l} = \frac{60\,EI_0}{\rho A_0 l^4}$$

$$\omega = 7.7460\sqrt{\frac{EI_0}{\rho A_0 l^4}}$$

8.62 Let $W(x) = \frac{c_0}{24EI}(x^4 - 4lx^3 + 6l^2x^2)$



$$\frac{d^2W}{dx^2} = \frac{c_0}{2EI}(x^2 - 2lx + l^2)$$

$$V_{max} = \frac{EI}{2}\int_0^l\left(\frac{d^2W}{dx^2}\right)^2 dx + \frac{1}{2}k\,[W(x=l)]^2 + \frac{1}{2}k_t\left[\frac{dW}{dx}(x=l)\right]^2$$

Since $\frac{k}{2}\left[W(x=\ell)\right]^2 = \frac{k}{2}\, C_0^2 \,\frac{\ell^8}{64\,E^2 I^2}$,

$$\frac{k_t}{2}\left[\frac{dW}{dx}(x=\ell)\right]^2 = \frac{k_t}{2}\, C_0^2\, \frac{\ell^6}{36 E^2 I^2},$$

and $\frac{1}{2} EI \int_0^\ell \left(\frac{d^2 W}{dx^2}\right)^2 dx = \frac{C_0^2\, \ell^5}{40\, EI}$ (from solution of problem 8.55)

$$V_{max} = \frac{C_0^2\, \ell^5}{40\, EI} + \frac{k\, C_0^2\, \ell^8}{128\, E^2 I^2} + \frac{k_t\, C_0^2\, \ell^6}{72\, E^2 I^2}$$

$$T_{max} = \frac{\omega^2 \rho A}{2}\int_0^\ell \left[W(x)\right]^2 dx = \omega^2 \rho A \left(\frac{13\, C_0^2\, \ell^9}{6480\, E^2 I^2}\right) \quad \text{from problem } 8.55$$

$V_{max} = T_{max}$ gives

$$\omega^2 = \left(12.4615\,\frac{EI}{\rho A \ell^4} + 3.8942\,\frac{k}{\rho A \ell} + 6.9231\,\frac{k_t}{\rho A \ell^4}\right)^{1/2}$$

8.63 $W(x) = C_1\left(1 - \cos\frac{2\pi x}{\ell}\right)$, $\quad \frac{d^2 W}{dx^2} = C_1\left(\frac{2\pi}{\ell}\right)^2 \cos\frac{2\pi x}{\ell}$

$$T_{max} = \frac{\omega^2}{2}\int_0^\ell \rho A\left[W(x)\right]^2 dx = \frac{\omega^2 \rho A\, C_1^2}{2}\int_0^\ell \left(1 + \cos^2\frac{2\pi x}{\ell} - 2\cos\frac{2\pi x}{\ell}\right)dx$$

$$= \frac{\rho A\,\omega^2 C_1^2}{2}\left[x + \left(\frac{x}{2} + \frac{\ell}{8\pi}\sin\frac{4\pi x}{\ell}\right) + \left(\frac{\ell}{\pi}\sin\frac{2\pi x}{\ell}\right)\right]_0^\ell$$

$$= \frac{3}{4}\rho A\,\omega^2 C_1^2\, \ell$$

$$V_{max} = \frac{EI}{2}\int_0^\ell \left(\frac{d^2 W}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^\ell C_1^2\left(\frac{2\pi}{\ell}\right)^4 \cos^2\frac{2\pi x}{\ell}\, dx$$

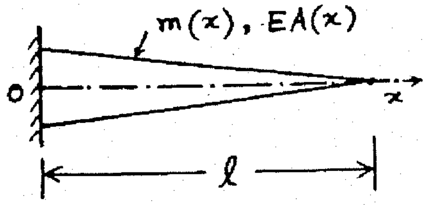
$$= \frac{8 EI\, C_1^2 \pi^4}{\ell^4}\left(\frac{x}{2} + \frac{\ell}{8\pi}\sin\frac{4\pi x}{\ell}\right)_0^\ell = \frac{4 EI\, C_1^2 \pi^4}{\ell^3}$$

$T_{max} = V_{max}$ gives

$$\omega^2 = \frac{4 EI\, C_1^2 \pi^4}{\ell^3}\cdot\frac{4}{3\rho A\, C_1^2\, \ell} = \frac{519.52\, EI}{\rho A \ell^4}$$

$$\omega = 22.7930\sqrt{\frac{EI}{\rho A \ell^4}}$$

8.64 $U(x) = C_1 \sin\frac{\pi x}{2\ell}$



$$\frac{dU}{dx} = C_1\,\frac{\pi}{2\ell}\cos\frac{\pi x}{2\ell}$$

$$V_{max} = \frac{1}{2}\int_0^\ell EA(x)\left(\frac{dU}{dx}\right)^2 dx$$

$$= \frac{1}{2} \int_0^{l} 2EA_0\left(1-\frac{x}{l}\right) c_1^2 \left(\frac{\pi}{2l}\right)^2 \cos^2 \frac{\pi x}{2l} \cdot dx$$

$$= EA_0\, c_1^2 \left(\frac{\pi}{2l}\right)^2 \left[\int_0^{l} \cos^2 \frac{\pi x}{2l}\, dx - \int_0^{l} \frac{x}{l} \cos^2 \frac{\pi x}{2l}\, dx \right]$$

$$= EA_0\, c_1^2 \left(\frac{\pi}{2l}\right)^2 \left[\left(\frac{x}{2} + \frac{l}{2\pi} \sin \frac{\pi x}{l}\right)_0^{l} - \frac{1}{l}\left(\frac{x^2}{4} + \frac{x \sin \frac{\pi x}{l}}{\left(\frac{2\pi}{l}\right)} + \frac{\cos \frac{\pi x}{l}}{\left(\frac{8\pi^2}{4l^2}\right)}\right)_0^{l} \right]$$

$$= \frac{EA_0\, c_1^2 \pi^2}{16\, l} \left(1 + \frac{4}{\pi^2}\right)$$

$$T_{max} = \frac{\omega^2}{2} \int_0^{l} m(x)\, U^2\, dx = \frac{\omega^2}{2} \int_0^{l} 2 m_0 \left(1-\frac{x}{l}\right) c_1^2 \sin^2 \frac{\pi x}{2l} \cdot dx$$

$$= \omega^2 m_0\, c_1^2 \left[\int_0^{l} \sin^2 \frac{\pi x}{2l} \cdot dx - \int_0^{l} \frac{x}{l} \sin^2 \frac{\pi x}{2l}\, dx \right]$$

$$= \omega^2 m_0\, c_1^2 \left[\left(\frac{x}{2} - \frac{l}{2\pi} \sin \frac{\pi x}{l}\right)_0^{l} - \frac{1}{l}\left(\frac{x^2}{4} - \frac{x \sin \frac{\pi x}{l}}{\left(\frac{2\pi}{l}\right)} - \frac{\cos \frac{\pi x}{l}}{\left(\frac{8\pi^2}{4l^2}\right)}\right)_0^{l} \right]$$

$$= \omega^2 m_0\, c_1^2 \frac{l}{4} \left(1 - \frac{4}{\pi^2}\right)$$

$V_{max} = T_{max}$ gives

$$\omega^2 = \frac{EA_0 \pi^2}{16\, l} \frac{\left(1 + \frac{4}{\pi^2}\right) 4}{m_0 l \left(1 - \frac{4}{\pi^2}\right)} = 5.8303 \frac{EA_0}{m_0\, l^2}$$

$$\therefore \omega_1 = 2.4146 \sqrt{\frac{EA_0}{m_0\, l^2}}$$

8.65 We take the deflection curve satisfying the boundary conditions

$w(0, y) = w(a, y) = w(x, 0) = w(x, b) = 0$ as

$$w(x, y) = c_1\, x y\, (x-a)(y-b)$$

$$\frac{\partial w}{\partial x} = c_1\, y\, (x-a)(y-b) + c_1\, xy\, (y-b)$$

$$\frac{\partial w}{\partial y} = c_1\, x\, (x-a)(y-b) + c_1\, xy\, (x-a)$$

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = c_1^2\, y^2 (y-b)^2 (x-a)^2 + c_1^2\, x^2 y^2 (y-b)^2 + c_1^2\, x^2 (x-a)^2 (y-b)^2 + c_1^2\, x^2 y^2 (x-a)^2$$

$$= c_1^2 \left[2x^2 y^4 + x^2 y^2 (2a^2 + 2b^2) + x^2 y^3 (-4b) + y^4 (a^2) + y^2 (a^2 b^2) + y^3 (-2ba^2) + xy^4 (-2a) + xy^2 (-2ab^2) + xy^3 (4ab) + 2x^4 y^2 + x^4 (b^2) + x^4 y\, (-2b) + x^2 (a^2 b^2) + x^2 y (-2ba^2) + x^3 y^2 (-2a) + x^3 (-2ab^2) + x^3 y\, (4ab) + x^3 y^2 (-2a) \right]$$

$$\int_0^a \int_0^b \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dx\ dy = c_1^2 \left(\frac{a^3 b^5}{45} + \frac{a^5 b^3}{45}\right)$$

$$w^2(x,y) = c_1^2 \left[x^4 y^4 + a^2 x^2 y^4 + b^2 x^4 y^2 + a^2 b^2 x^2 y^2 - 2a\ x^3 y^4 - 2b\, x^4 y^3 + 2ab\, x^3 y^3 + 2ab\, x^3 y^3 - 2a^2 b\, x^2 y^3 - 2ab^2 x^3 y^2\right]$$

$$\int_0^a \int_0^b w^2 \cdot dx\ dy = c_1^2\ a^5 b^5 / 900$$

$$V_{max} = \frac{P}{2} \frac{c_1^2}{45} (a^3 b^3)(a^2 + b^2)$$

$$T_{max} = \frac{\omega^2 \rho}{2} \cdot \frac{c_1^2\ a^5 b^5}{900}$$

$V_{max} = T_{max}$ gives $\dfrac{\omega^2 \rho}{P} = 20\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right)$

$$\therefore\ \omega_1 = 4.4721 \sqrt{\frac{P}{\rho}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}$$

Exact value of $\omega_1 = \pi \sqrt{\dfrac{P}{\rho}\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right)}$

8.66 For a shaft under torsion, the shear stress (τ) induced at a radius r from the center of the cross section is given by

$$\tau = \frac{M_t(x)\ r}{J} \tag{1}$$

where $M_t(x)$ = torque at section x. The potential energy of the shaft is given by the strain energy:

$$V = \frac{1}{2} \int \frac{1}{G} \tau^2\ dA\ dx \tag{2}$$

Using Eq. (8.61):

$$M_t(x) = GJ\ \frac{\partial \theta}{\partial x} \tag{3}$$

Eq. (2) can be expressed as

$$V = \frac{1}{2} \int_0^\ell GJ \left(\frac{\partial \theta}{\partial x}\right)^2 dx \tag{4}$$

The kinetic energy of the shaft can be written as

$$T = \frac{1}{2} \int_0^\ell \rho\ J \left(\frac{\partial \theta}{\partial t}\right)^2 dx \tag{5}$$

By assuming a harmonic variation of $\theta(x,t)$ as

$$\theta(x,\ t) = \Theta(x) \cos \omega\, t \tag{6}$$

and equating V_{max} to T_{max}, we obtain

$$\omega^2 = \frac{\int_0^\ell GJ \left(\frac{d\Theta}{dx}\right)^2 dx}{\int_0^\ell \rho\, J\, \Theta^2\, dx} \tag{7}$$

For the shaft shown in Fig. 8.36,

$$G = 80\ (10^9)\ N/m^2 \quad ; \quad \rho\, g = 76.5\ kN/m^3$$

$$J = \frac{\pi}{32}\, d^4 = \frac{\pi}{32}\, (0.05^4) = 61.3594\ (10^{-8})\ m^4$$

Let $\Theta(x)$ be assumed to vary linearly on either side of the disc for simplicity:

$$\begin{aligned} \Theta(x) &= \frac{\theta_0\, x}{0.8} \quad ; \quad 0 \le x \le 0.8 \\ &= \frac{\theta_0\, (1 - x)}{0.2} \quad ; \quad 0.8 \le x \le 1 \end{aligned} \tag{8}$$

where θ_0 is the angular displacement of the disc. Equation (8) satisfies the boundary conditions and gives:

$$\begin{aligned} \frac{d\Theta}{dx} &= \frac{\theta_0}{0.8} \quad ; \quad 0 \le x \le 0.8 \\ &= -\frac{\theta_0}{0.2} \quad ; \quad 0.8 \le x \le 1.0 \end{aligned} \tag{9}$$

$$\int_0^\ell GJ \left(\frac{d\Theta}{dx}\right)^2 dx = 306\ 797\ \theta_0^2 \quad ; \quad \int_0^\ell \rho\, J\, \Theta^2\, dx = 159.5384\ (10^{-5})\ \theta_0^2$$

Thus Eq. (7) gives the approximate natural frequency as:

$$\omega^2 \approx \frac{306797\ \theta_0^2}{159.5384\ (10^{-5})\ \theta_0^2} = 1923.0292\ (10^5)$$

$$\omega \approx 13{,}867.3328\ rad/sec$$

For comparison, the torsional natural frequency of a fixed-fixed shaft with no intermediate disc, is given by (see Fig. 8.12):

$$\omega_1 = \frac{\pi\, c}{\ell} = \frac{\pi}{\ell} \sqrt{\frac{G}{\rho}} = \frac{\pi}{1.0} \sqrt{\frac{80\ (10^9)\ (9.81)}{76500}} = 10{,}062.3557\ rad/sec$$

8.67 (a) $W(x) = c_1 x (l - x)$, Let P = tension

$$V_{max} = \frac{1}{2}\int_0^l P\left(\frac{dw}{dx}\right)^2 dx = \frac{1}{2} c_1^2 P \int_0^l (l-2x)^2 dx$$

$$= \frac{1}{2} P c_1^2 \left[l^2 x + \frac{4}{3} x^3 - 2 l x^2 \right]_0^l = \frac{1}{6} P c_1^2 l^3$$

$$T_{max} = \frac{\omega^2}{2}\int_0^l \rho [W]^2 dx = \frac{1}{2}\rho c_1^2 \omega^2 \int_0^l (x^2 l^2 + x^4 - 2 l x^3) dx$$

$$= \frac{1}{2}\rho c_1^2 \omega^2 \left[\frac{1}{3} x^3 l^2 + \frac{1}{5} x^5 - \frac{1}{2} l x^4 \right]_0^l = \frac{1}{60}\omega^2 \rho c_1^2 l^5$$

$V_{max} = T_{max}$ gives

$$\omega = \left(\frac{10 P}{\rho l^2}\right)^{1/2} = 3.1623 \sqrt{\frac{P}{\rho l^2}}$$

Exact value is $\omega_1 = \pi \sqrt{\frac{P}{\rho l^2}}$.

(b) $W(x) = c_1 x (l - x) + c_2 x^2 (l - x)^2$

$$\frac{dW}{dx} = c_1 (l - 2x) + c_2 (2 l^2 x + 4 x^3 - 6 l x^2)$$

$$V_{max} = \frac{1}{2}\int_0^l P\left(\frac{dw}{dx}\right)^2 dx = \frac{P}{2}\int_0^l \left[c_1^2 (l^2 + 4x^2 - 4 l x)\right.$$

$$+ c_2^2 (4 l^4 x^2 + 16 x^6 + 36 l^2 x^4 + 16 l^2 x^4 - 48 l x^5 - 24 l^3 x^3)$$

$$\left. + 2 c_1 c_2 (2 l^3 x + 4 l x^3 - 6 l^2 x^2 - 4 l^2 x^2 - 8 x^4 + 12 l x^3)\right] dx$$

$$= \frac{P}{2}\left(\frac{1}{3} c_1^2 l^3 + \frac{2}{105} c_2^2 l^7 + \frac{2}{15} c_1 c_2 l^5\right)$$

$$T_{max} = \frac{\omega^2}{2}\int_0^l \rho [W(x)]^2 dx = \frac{\rho\omega^2}{2}\int_0^l \left[c_1^2 (l^2 x^2 + x^4 - 2 l x^3)\right.$$

$$+ c_2^2 (l^4 x^4 + x^8 + 6 l^2 x^6 - 4 l x^7 - 4 l^3 x^5)$$

$$\left. + c_1 c_2 (2 l^3 x^3 + 6 l x^5 - 6 l^2 x^4 - 2 x^6)\right] dx$$

$$= \frac{\rho\omega^2}{2}\left[\frac{1}{30} c_1^2 l^5 + \frac{1}{630} c_2^2 l^9 + \frac{1}{70} c_1 c_2 l^7\right]$$

$$X = P\left(\frac{1}{3} c_1^2 l^3 + \frac{2}{105} c_2^2 l^7 + \frac{2}{15} c_1 c_2 l^5\right)$$

$$Y = \rho\left(\frac{1}{30} c_1^2 l^5 + \frac{1}{630} c_2^2 l^9 + \frac{1}{70} c_1 c_2 l^7\right)$$

$$\frac{\partial X}{\partial c_1} = P\left(\frac{2}{3} c_1 l^3 + \frac{2}{15} c_2 l^5\right), \quad \frac{\partial X}{\partial c_2} = P\left(\frac{4}{105} c_2 l^7 + \frac{2}{15} c_1 l^5\right)$$

$$\frac{\partial Y}{\partial c_1} = \rho\left(\frac{1}{15} c_1 l^5 + \frac{1}{70} c_2 l^7\right), \quad \frac{\partial Y}{\partial c_2} = \rho\left(\frac{1}{315} c_2 l^9 + \frac{1}{70} c_1 l^7\right)$$

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0, \quad \frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \quad \text{give}$$

$$P\left(c_1 \frac{2}{3}l^3 + c_2 \frac{2}{15} l^5\right) - \omega^2 \rho \left(c_1 \frac{l^5}{15} + c_2 \frac{l^7}{70}\right) = 0$$

$$P\left(c_2 \frac{4l^7}{105} + c_1 \frac{2l^5}{15}\right) - \omega^2 \rho \left(c_2 \frac{l^9}{315} + c_1 \frac{l^7}{70}\right) = 0$$

i.e., $$c_1\left(\frac{2}{3} P l^3 - \frac{\rho\omega^2 l^5}{15}\right) + c_2\left(\frac{2}{15} P l^5 - \frac{\rho \omega^2 l^7}{70}\right) = 0$$

$$c_1\left(\frac{2}{15} P l^5 - \frac{\rho\omega^2 l^7}{70}\right) + c_2\left(\frac{4}{105} P l^7 - \frac{\rho \omega^2 l^9}{315}\right) = 0$$

Setting the determinant of the coefficients of c_1 and c_2 to zero gives

$$\left(\frac{2}{3} P l^3 - \frac{1}{15}\rho\omega^2 l^5\right)\left(\frac{4}{105} P l^7 - \frac{1}{315}\rho\omega^2 l^9\right)$$

$$- \left(\frac{2}{15} P l^5 - \frac{1}{70}\rho\omega^2 l^7\right)\left(\frac{2}{15} P l^5 - \frac{1}{70}\rho\omega^2 l^7\right) = 0$$

i.e., $$\omega^4 - 112\left(\frac{P}{\rho l^2}\right)\omega^2 + 1008\left(\frac{P}{\rho l^2}\right)^2 = 0$$

$$\omega^2 = 9.8697 \frac{P}{\rho l^2}, \quad 102.1302 \frac{P}{\rho l^2}$$

$$\therefore \quad \omega_1 = 3.1416\sqrt{\frac{P}{\rho l^2}}, \qquad \omega_2 = 10.1059\sqrt{\frac{P}{\rho l^2}}$$

First mode Third mode

Exact solution is :

$$\omega_1 = 3.1416\sqrt{\frac{P}{\rho l^2}}, \quad \omega_2 = 6.2832\sqrt{\frac{P}{\rho l^2}}, \quad \omega_3 = 9.4248\sqrt{\frac{P}{\rho l^2}}$$

8.68 (a)

$$U(x) = \frac{c_1}{l} x$$
$$\frac{dU}{dx} = \frac{c_1}{l}$$

$$V_{max} = \frac{1}{2}\int_0^l EA\left(\frac{dU}{dx}\right)^2 dx$$
$$= \frac{1}{2}\int_0^l EA\left(\frac{c_1}{l}\right)^2 dx$$
$$= \frac{EA c_1^2}{2l} \quad \text{for uniform beam}$$

$$T_{max} = \frac{\rho\omega^2}{2}\int_0^l A\, U^2 dx = \frac{\rho\omega^2}{2}\int_0^l A \frac{c_1^2}{l^2} x^2 dx$$
$$= \frac{\omega^2 \rho A c_1^2 l}{6} \quad \text{for uniform beam}$$

$T_{max} = V_{max}$ gives $\omega^2 = \frac{3E}{\rho l^2}$

$$\therefore \omega_1 = 1.73205\sqrt{\frac{E}{\rho l^2}}$$

(b) $V_{max} = \frac{EA}{2} \int_0^l \left(\frac{dU}{dx}\right)^2 dx$

$= \frac{EA}{2} \left[\frac{c_1^2}{l} + \frac{4}{3} \frac{c_2^2}{l} + \frac{2 c_1 c_2}{l} \right]$

$T_{max} = \frac{\rho \omega^2}{2} \int_0^l A \cdot U^2 . dx$

$= \frac{\rho A \omega^2}{2} \left[\frac{1}{3} c_1^2 l + \frac{1}{5} c_2^2 l + \frac{1}{2} c_1 c_2 l \right]$

$U(x) = \frac{c_1 x}{l} + \frac{c_2 x^2}{l^2}$

$\frac{dU}{dx} = \frac{c_1}{l} + \frac{2 c_2 x}{l^2}$

Eqs. (E_{11}) and (E_{12}) of Example 8.12 give

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 \quad \text{and} \quad \frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \qquad (E.1)$$

where $$X = \frac{EA}{2l} \left[c_1^2 + \frac{4}{3} c_2^2 + 2 c_1 c_2 \right] \qquad (E.2)$$

and $$Y = \frac{\rho A l}{2} \left[\frac{1}{3} c_1^2 + \frac{1}{5} c_2^2 + \frac{1}{2} c_1 c_2 \right] \qquad (E.3)$$

Eqs. (E.1) to (E.3) give

$$c_1 \left(\frac{EA}{l} - \frac{1}{3} \omega^2 \rho A l \right) + c_2 \left(\frac{EA}{l} - \frac{1}{4} \omega^2 \rho A l \right) = 0$$

$$c_2 \left(\frac{4}{3} \frac{EA}{l} - \frac{1}{5} \omega^2 \rho A l \right) + c_1 \left(\frac{EA}{l} - \frac{1}{4} \omega^2 \rho A l \right) = 0$$

Frequency equation is

$$\begin{vmatrix} (1 - \frac{1}{3} \lambda) & (1 - \frac{1}{4} \lambda) \\ (1 - \frac{1}{4} \lambda) & (\frac{4}{3} - \frac{1}{5} \lambda) \end{vmatrix} = \frac{\lambda^2}{240} - \frac{13 \lambda}{90} + \frac{1}{3} = 0$$

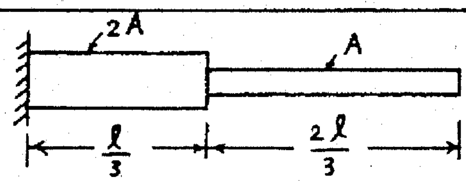
where $\lambda = \frac{\omega^2 \rho l^2}{E}$

$\lambda = 2.48596, \quad 32.1807$

$\therefore \omega_1 = 1.57669 \sqrt{\frac{E}{\rho l^2}}, \quad \omega_2 = 5.67280 \sqrt{\frac{E}{\rho l^2}}$

8.69

$U(x) = c_1 \sin \frac{\pi x}{2l} + c_2 \sin \frac{3 \pi x}{2l}$



$\frac{dU}{dx} = \frac{\pi}{2l} \left(c_1 \cos \frac{\pi x}{2l} + 3 c_2 \cos \frac{3 \pi x}{2l} \right)$

$$V_{max} = \frac{EA}{2} \left(\frac{\pi}{2l} \right)^2 \left[\int_0^{l/3} 2 \left(c_1 \cos \frac{\pi x}{2l} + 3 c_2 \cos \frac{3 \pi x}{2l} \right)^2 dx + \int_{l/3}^{l} \left(c_1 \cos \frac{\pi x}{2l} + 3 c_2 \cos \frac{3 \pi x}{2l} \right)^2 dx \right]$$

Using the relations $\int \cos^2 \frac{\pi x}{2l} dx = \frac{x}{2} + \frac{l}{2\pi} \sin \frac{\pi x}{l}$,

$$\int \cos^2 \frac{3\pi x}{2l}\, dx = \frac{x}{2} + \frac{l}{6\pi} \sin \frac{3\pi x}{l},$$

$$\int \cos \frac{\pi x}{2l} \cdot \cos \frac{3\pi x}{2l} \cdot dx = \frac{\sin \frac{\pi x}{l}}{(2\pi/l)} + \frac{\sin \frac{2\pi x}{l}}{(4\pi/l)},$$

$$V_{max} = \frac{EA\pi^2}{8l^2}\Big[c_1^2\left(\tfrac{2}{3}l + \frac{l}{2\pi}\sin\frac{\pi}{3}\right) + c_2^2\,(6l) + c_1c_2\left(\frac{3l}{\pi}\sin\frac{\pi}{3} + \frac{3l}{2\pi}\sin\frac{2\pi}{3}\right)\Big]$$

$$T_{max} = \frac{\rho\omega^2}{2}\Big[\int_0^{l/3} 2A\Big\{c_1^2 \sin^2\frac{\pi x}{2l} + c_2^2 \sin^2\frac{3\pi x}{2l} + 2c_1c_2 \sin\frac{\pi x}{2l}\cdot\sin\frac{3\pi x}{2l}\Big\}dx + \int_{l/3}^{l} A\Big\{c_1^2\sin^2\frac{\pi x}{2l} + c_2^2\sin^2\frac{3\pi x}{2l} + 2c_1c_2\sin\frac{\pi x}{2l}\sin\frac{3\pi x}{2l}\Big\}dx\Big]$$

But $\displaystyle\int \sin^2 \frac{\pi x}{2l}\, dx = \frac{x}{2} - \frac{l}{2\pi}\sin\frac{\pi x}{l},$

$$\int \sin^2\frac{3\pi x}{2l}\, dx = \frac{x}{2} - \frac{l}{6\pi}\sin\frac{3\pi x}{l},$$

$$\int \sin\frac{\pi x}{2l}\cdot\sin\frac{3\pi x}{2l}\, dx = \frac{\sin\frac{\pi x}{l}}{(2\pi/l)} - \frac{\sin\frac{2\pi x}{l}}{(4\pi/l)}$$

$$T_{max} = \frac{\rho A\omega^2}{2}\Big[c_1^2\left(\frac{2l}{3} - \frac{l}{2\pi}\sin\frac{\pi}{3}\right) + c_2^2\left(\frac{2l}{3}\right) + c_1c_2\left(\frac{l}{\pi}\sin\frac{\pi}{3} - \frac{l}{2\pi}\sin\frac{2\pi}{3}\right)\Big]$$

$$X = \frac{EA\pi^2}{4l^2}\Big[c_1^2\left(\frac{2l}{3} + \frac{l}{2\pi}\sin\frac{\pi}{3}\right) + c_2^2\,(6l) + c_1c_2\left(\frac{3l}{\pi}\sin\frac{\pi}{3} + \frac{3l}{2\pi}\sin\frac{2\pi}{3}\right)\Big]$$

$$Y = \rho A\Big[c_1^2\left(\frac{2l}{3} - \frac{l}{2\pi}\sin\frac{\pi}{3}\right) + c_2^2\left(\frac{2l}{3}\right) + c_1c_2\left(\frac{l}{\pi}\sin\frac{\pi}{3} - \frac{l}{2\pi}\sin\frac{2\pi}{3}\right)\Big]$$

$$\frac{\partial X}{\partial c_1} = \frac{EA\pi^2}{4l^2}\Big[2c_1\left(\frac{2l}{3} + \frac{l}{2\pi}\sin\frac{\pi}{3}\right) + c_2\left(\frac{3l}{\pi}\sin\frac{\pi}{3} + \frac{3l}{2\pi}\sin\frac{2\pi}{3}\right)\Big]$$

$$= \frac{EA\pi^2}{4l^2}\big[2c_1(0.8045l) + c_2(1.2405\,l)\big]$$

$$\frac{\partial X}{\partial c_2} = \frac{EA\pi^2}{4l^2}\Big[c_2(12l) + c_1\left(\frac{3l}{\pi}\sin\frac{\pi}{3} + \frac{3l}{2\pi}\sin\frac{2\pi}{3}\right)\Big]$$

$$= \frac{EA\pi^2}{4l^2}\big[c_2(12l) + c_1(1.2405\,l)\big]$$

$$\frac{\partial Y}{\partial c_1} = \rho A\Big[2c_1\left(\frac{2l}{3} - \frac{l}{2\pi}\sin\frac{\pi}{3}\right) + c_2\left(\frac{l}{\pi}\sin\frac{\pi}{3} - \frac{l}{2\pi}\sin\frac{2\pi}{3}\right)\Big]$$

$$= \rho A\big[2c_1(0.5288\,l) + c_2(0.1378\,l)\big]$$

$$\frac{\partial Y}{\partial c_2} = \rho A\Big[c_2\left(\frac{4l}{3}\right) + c_1\left(\frac{l}{\pi}\sin\frac{\pi}{3} - \frac{l}{2\pi}\sin\frac{2\pi}{3}\right)\Big]$$

$$= \rho A\big[c_2(1.3333\,l) + c_1(0.1378\,l)\big]$$

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 \quad \text{and} \quad \frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \quad \text{give}$$

$$c_1\left(\frac{EA\pi^2}{l}\, 0.40225 - \rho A\omega^2 l\, 1.0576\right) + c_2\left(\frac{EA\pi^2}{l}\, 0.31012 - \rho A\omega^2 l\, 0.1378\right) = 0$$

$$c_1\left(\frac{EA\pi^2}{l}\, 0.31012 - \rho A\omega^2 l\, 0.1380\right) + c_2\left(\frac{EA\pi^2}{l}\, 3.0 - \rho A\omega^2 l\, 1.3333\right) = 0$$

Frequency equation is

$$\left(0.40225\, \frac{EA\pi^2}{l} - \rho A\omega^2 l\, 1.0576\right)\left(\frac{EA\pi^2}{l}\, 3.0 - \rho A\omega^2 l\, 1.3333\right) - \left(\frac{EA\pi^2}{l}\, 0.31012 - \rho A\omega^2 l\, 0.1380\right)^2 = 0$$

i.e.,
$$\omega^4 - 25.7090\, \omega^2 \left(\frac{E}{\rho l^2}\right) + 77.7692\left(\frac{E}{\rho l^2}\right)^2 = 0$$

$$\omega^2 = \left(\frac{25.7090 \pm 18.7010}{2}\right)\frac{E}{\rho l^2}$$

$$\therefore\ \omega_1 = 1.8719\sqrt{\frac{E}{\rho l^2}}\,, \qquad \omega_2 = 4.7022\sqrt{\frac{E}{\rho l^2}}$$

8.70 $U(x) = c_1 \sin\frac{\pi x}{2l} + c_2 \sin\frac{3\pi x}{2l}\,, \quad \frac{dU}{dx} = \frac{\pi}{2l}\left(c_1 \cos\frac{\pi x}{2l} + 3c_2 \cos\frac{3\pi x}{2l}\right)$

$$V_{max} = \frac{1}{2}\int_0^l EA(x)\left(\frac{dU}{dx}\right)^2 dx = \frac{1}{2}\int_0^l 2EA_0\left(1-\frac{x}{l}\right)\left(\frac{\pi}{2l}\right)^2\left(c_1^2 \cos^2\frac{\pi x}{2l} + 9c_2^2\cos^2\frac{3\pi x}{2l} + 6c_1c_2 \cos\frac{\pi x}{2l}\cos\frac{3\pi x}{2l}\right)dx$$

$$= c_1^2\,\frac{EA_0\pi^2}{16l}\left(1+\frac{4}{\pi^2}\right) + c_2^2\,\frac{9EA_0\pi^2}{16l}\left(1+\frac{4}{9\pi^2}\right) + c_1c_2\,\frac{3EA_0\pi^2}{2l}$$

$$T_{max} = \frac{\omega^2}{2}\int_0^l m(x)\cdot U^2\cdot dx = m_0\omega^2\int_0^l\left(1-\frac{x}{l}\right)\left(c_1^2 \sin^2\frac{\pi x}{2l} + c_2^2 \sin^2\frac{3\pi x}{2l} + 2c_1c_2 \sin\frac{\pi x}{2l}\sin\frac{3\pi x}{2l}\right)dx$$

$$= c_1^2\,\frac{\omega^2 m_0 l}{4}\left(1-\frac{4}{\pi^2}\right) + c_2^2\,\frac{\omega^2 m_0 l}{4}\left(1-\frac{4}{9\pi^2}\right) + c_1c_2\,\frac{2\omega^2 m_0 l}{\pi^2}$$

$$X = c_1^2\,\frac{EA_0}{l}(0.86685) + c_2^2\,\frac{EA_0}{l}(5.80165) + c_1c_2\,\frac{EA_0}{l}(1.5)$$

$$Y = c_1^2\, m_0 l\,(0.14868) + c_2^2\, m_0 l\,(0.23874) + c_1c_2\, m_0 l\,(0.20264)$$

$$\frac{\partial X}{\partial c_1} = c_1\,\frac{EA_0}{l}(1.73370) + c_2\,\frac{EA_0}{l}(1.5)$$

$$\frac{\partial X}{\partial c_2} = c_1 \frac{EA_0}{l}(1.5) + c_2 \frac{EA_0}{l}(11.60330)$$

$$\frac{\partial Y}{\partial c_1} = c_1\, m_0 l\,(0.29736) + c_2\, m_0 l\,(0.20264)$$

$$\frac{\partial Y}{\partial c_2} = c_1\, m_0 l\,(0.20246) + c_2\, m_0 l\,(0.47748)$$

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 \quad \text{and} \quad \frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \quad \text{give}$$

$$c_1\left(\frac{EA_0}{l}\, 1.73370 - \omega^2 m_0 l\, 0.29736\right) + c_2\left(\frac{EA_0}{l}\, 1.5 - \omega^2 m_0 l\, 0.20264\right) = 0$$

$$c_1\left(\frac{EA_0}{l}\, 1.5 - \omega^2 m_0 l\, 0.20246\right) + c_2\left(\frac{EA_0}{l}\, 11.60330 - \omega^2 m_0 l\, 0.47748\right) = 0$$

Frequency equation is

$$\left(\frac{EA_0}{l}\, 1.7337 - \omega^2 m_0 l\, 0.29736\right)\left(\frac{EA_0}{l}\, 11.6033 - \omega^2 m_0 l\, 0.47748\right) - \left(\frac{EA_0}{l}\, 1.5 - \omega^2 m_0 l\, 0.20264\right)^2 = 0$$

or $$\omega^4 - 36.369\, \omega^2 \left(\frac{EA_0}{m_0 l^2}\right) + 177.04292 \left(\frac{EA_0}{m_0 l^2}\right)^2 = 0$$

$$\omega^2 = \left(\frac{36.369 \pm 24.7898}{2}\right) \frac{EA_0}{m_0 l^2}$$

$$\therefore\ \omega_1 = 2.4062 \sqrt{\frac{EA_0}{m_0 l^2}}, \qquad \omega_2 = 5.5299 \sqrt{\frac{EA_0}{m_0 l^2}}$$

8.71 For a string, $V_{max} = \frac{1}{2}\int_0^l P\left(\frac{dW}{dx}\right)^2 \cdot dx$, $T_{max} = \frac{\omega^2}{2}\int_0^l \rho\, W^2 \cdot dx$

Here $W(x) = c_1 x (l - x) + c_2 x^2 (l - x)^2$

$$\frac{dW}{dx} = c_1 l - 2c_1 x + 2c_2 l^2 x - 6c_2 l x^2 + 4c_2 x^3$$

$$\int_0^l \left(\frac{dW}{dx}\right)^2 dx = c_1^2 \frac{l^3}{3} + c_2^2 \frac{2}{105} l^7 + c_1 c_2 \frac{2}{15} l^5$$

$$\int_0^l W^2\, dx = c_1^2 \frac{l^5}{30} + c_2^2 \frac{l^9}{630} + c_1 c_2 \frac{l^7}{70}$$

Equating T_{max} and V_{max}, we get $\omega^2 = \frac{X}{Y}$

where $$X = \frac{P}{2}\left[c_1^2 \frac{l^3}{3} + c_2^2 \frac{2}{105} l^7 + c_1 c_2 \frac{2}{15} l^5\right]$$

and $$Y = \frac{\rho}{2}\left[c_1^2 \frac{l^5}{30} + c_2^2 \frac{l^9}{630} + c_1 c_2 \frac{l^7}{70}\right]$$

$\frac{\partial(\omega^2)}{\partial c_1} = \frac{\partial(\omega^2)}{\partial c_2} = 0$ yield Eqs. (E_{11}) and (E_{12}) of Example 8.13.

$$\frac{\partial X}{\partial c_1} = \frac{P}{2}\left[c_1 \frac{2}{3}\ell^3 + c_2 \frac{2}{15}\ell^5\right], \quad \frac{\partial Y}{\partial c_1} = \frac{\rho}{2}\left[c_1 \frac{\ell^5}{15} + c_2 \frac{\ell^7}{70}\right]$$

$$\frac{\partial X}{\partial c_2} = \frac{P}{2}\left[c_1 \frac{2}{15}\ell^5 + c_2 \frac{4}{105}\ell^7\right], \quad \frac{\partial Y}{\partial c_2} = \frac{\rho}{2}\left[c_1 \frac{\ell^7}{70} + c_2 \frac{\ell^9}{315}\right]$$

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 \Rightarrow c_1\left(\frac{P\ell^3}{3} - \frac{\rho\omega^2\ell^5}{30}\right) + c_2\left(\frac{P\ell^5}{15} - \frac{\rho\omega^2\ell^7}{140}\right) = 0$$

$$\frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \Rightarrow c_1\left(\frac{P\ell^5}{15} - \frac{\rho\omega^2\ell^7}{140}\right) + c_2\left(\frac{2P\ell^7}{105} - \frac{\rho\omega^2\ell^9}{630}\right) = 0$$

By setting the determinant of the coefficient matrix of c_1 and c_2 to zero, we get the frequency equation as

$$\begin{vmatrix} \left(\frac{1}{3} - \frac{\lambda}{30}\right) & \left(\frac{1}{15} - \frac{\lambda}{140}\right) \\ \left(\frac{1}{15} - \frac{\lambda}{140}\right) & \left(\frac{2}{105} - \frac{\lambda}{630}\right) \end{vmatrix} = \lambda^2 - 112\lambda + 1008 = 0$$

where $\lambda = \dfrac{\rho\ell^2\omega^2}{P}$.

$$\lambda_1 = 9.8722, \quad \lambda_2 = 102.4144$$

$$\therefore \omega_1 = 3.142\sqrt{\frac{P}{\rho\ell^2}}, \quad \omega_2 = 10.12\sqrt{\frac{P}{\rho\ell^2}}$$

8.72

```
C==========================================================================
C
C PROGRAM
C MAIN PROGRAM FOR CALLING THE SUBROUTINE NONEQN
C
C==========================================================================
C FOLLOWING LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(2)
      COMMON /ABLOCK/ BETA
      DATA N,NINT,ITER,EPS/2,50,100,0.000001/
      DO 7 I=1,2
      IF (I .EQ. 1) BETA=0.01
      IF (I .EQ. 2) BETA=1.0
      WRITE (12,8) BETA
 8    FORMAT (//,2X,7H BETA =,E15.6,/)
      XS=0.001
      XINC=0.1
C END OF PROBLEM-DEPENDENT DATA
      WRITE (12,10) N,XS,XINC,NINT,ITER,EPS
 10   FORMAT (//,28H ROOTS OF NONLINEAR EQUATION,//,6H DATA:,/,
     2 2H N,4X,2H =,I4,/,3H XS,3X,2H =,E15.6,/,8H XINC  =,E15.6,/,
     3 8H NINT   =,I4,/,8H ITER   =,I4,/,8H EPS    =,E15.6)
      CALL NONEQN (N,X,XS,XINC,NINT,ITER,EPS)
      WRITE (12,20) (X(II),II=1,N)
 20   FORMAT (//,7H ROOTS:,/,(E15.6))
 7    CONTINUE
      STOP
      END
```

```
C======================================================================
C
C FUNCTION F(Y)
C THIS FUNCTION IS PROBLEM-DEPENDENT
C
C======================================================================
      FUNCTION F(Y)
      COMMON /ABLOCK/ BETA
      F=(Y/BETA)*TAN(Y)-1.0
      RETURN
      END
```

```
--------------------------------------------------------
   BETA =    0.100000E-01


ROOTS OF NONLINEAR EQUATION

DATA:
N      =    2
XS     =    0.100000E-02
XINC   =    0.100000E+00
NINT   =  50
ITER   = 100
EPS    =    0.100000E-05


ROOTS:
  0.998336E-01
  0.314477E+01


  BETA =    0.100000E+01


ROOTS OF NONLINEAR EQUATION

DATA:
N      =    2
XS     =    0.100000E-02
XINC   =    0.100000E+00
NINT   =  50
ITER   = 100
EPS    =    0.100000E-05


ROOTS:
  0.860334E+00
  0.342562E+01
```

8.73 Frequency equation for a fixed-fixed beam is (from Fig. 8.15):

$$\cos \beta l \cosh \beta l - 1 = 0$$

The main program, the function F(Y) and the output are given below.

```
C=======================================================================
C
C PROGRAM
C MAIN PROGRAM FOR CALLING THE SUBROUTINE NONEQN
C
C=======================================================================
C FOLLOWING LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(5)
      DATA N,NINT,ITER,EPS/5,50,100,0.000001/
      XS=2.0
      XINC=0.5
C END OF PROBLEM-DEPENDENT DATA
      WRITE (12,10) N,XS,XINC,NINT,ITER,EPS
 10   FORMAT (//,28H ROOTS OF NONLINEAR EQUATION,//,6H DATA:,/,
     2 2H N,4X,2H =,I4,/,3H XS,3X,2H =,E15.6,/,8H XINC   =,E15.6,/,
     3 8H NINT   =,I4,/,8H ITER   =,I4,/,8H EPS    =,E15.6)
      CALL NONEQN (N,X,XS,XINC,NINT,ITER,EPS)
      WRITE (12,20) (X(I1),I1=1,N)
 20   FORMAT (//,7H ROOTS:,/,(E15.6))
      STOP
      END
C=======================================================================
C
C FUNCTION F(Y)
C THIS FUNCTION IS PROBLEM-DEPENDENT
C
C=======================================================================
      FUNCTION F(Y)
      F=COS(Y)*COSH(Y)-1.0
      RETURN
      END
```

```
ROOTS OF NONLINEAR EQUATION

DATA:
N      =    5
XS     =    0.200000E+01
XINC   =    0.500000E+00
NINT   =   50
ITER   =  100
EPS    =    0.100000E-05


ROOTS:
  0.473004E+01
  0.785320E+01
  0.109956E+02
  0.141372E+02
  0.172788E+02
```

8.74 From Example 8.7, the n^{th} mode shape is given by

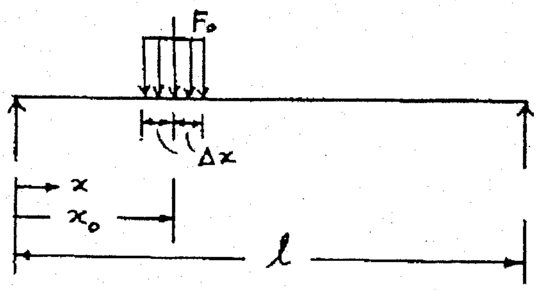
$$W_n(x) = C_{1n}\left\{(\cos\beta_n x - \cosh\beta_n x) - \left(\frac{\cos\beta_n l - \cosh\beta_n l}{\sin\beta_n l - \sinh\beta_n l}\right)(\sin\beta_n x - \sinh\beta_n x)\right\}$$

where $\beta_n l = 3.926602,\ 7.068583,\ 10.210176,\ 13.351768$ for $n = 1, 2, 3, 4$.
The program and results are given below.

```
C ====================================================================
C
C PROBLEM 8.74
C
C ====================================================================
      DIMENSION X(4),W(4,9)
      N=4
      M=9
      MD=10
C N       = NUMBER OF MODE SHAPES TO BE COMPUTED
C M       = NUMBER OF STATIONS IN THE BEAM AT WHICH VALUE OF DEFLECTION
C           IS REQUIRED
C MD      = NUMBGR OF SEGMENTS IN THE BEAM
C W(N,M) = MATRIX CONTAINING THE MODE SHAPES IN ROWS
      DATA X/3.926602,7.068583,10.210176,13.351768/
      DO 10 I=1,N
      DO 20 J=1,M
      Y=X(I)*REAL(J)/REAL(MD)
      YY=X(I)
 20   W(I,J)=(COS(Y)-COSH(Y))-((COS(YY)-COSH(YY))/(SIN(YY)-SINH(YY)))*
     2 (SIN(Y)-SINH(Y))
 10   CONTINUE
      WRITE (86,30)
 30   FORMAT (//,2X,43H MODE SHAPES OF FIXED-SIMPLY SUPPORTED BEAM,/)
      DO 40 I=1,N
 40   WRITE (86,50) I,(W(I,J),J=1,M)
 50   FORMAT (/,2X,6H MODE:,I4,4E15.6,/,(12X,4E15.6))
      STOP
      END
```

```
MODE SHAPES OF FIXED-SIMPLY SUPPORTED BEAM


MODE:    1  -0.133996E+00  -0.455738E+00  -0.848519E+00  -0.120675E+01
            -0.144486E+01  -0.150550E+01  -0.136498E+01  -0.103456E+01
            -0.557241E+00

MODE:    2  -0.382233E+00  -0.107449E+01  -0.149510E+01  -0.131923E+01
            -0.570351E+00   0.422676E+00   0.119881E+01   0.139349E+01
             0.917175E+00

MODE:    3  -0.690370E+00  -0.147476E+01  -0.112212E+01   0.204393E+00
             0.130049E+01   0.114192E+01  -0.111633E+00  -0.126025E+01
            -0.120557E+01

MODE:    4  -0.100204E+01  -0.141423E+01   0.927429E-01   0.139201E+01
             0.539917E+00  -0.114441E+01  -0.107568E+01   0.642578E+00
             0.137500E+01
```

8.75



Equation of motion is given by Eq. (8.77):

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = f(x,t) \qquad (E_1)$$

Representation of load:

Load is $f(x)$ which is zero everywhere except over a length $2\Delta x$ at $x = x_0$, where it is equal to $(F_0/2\Delta x)$. The load $f(x)$ can be expanded into Fourier sine series as

$$f(x) = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi x}{l} \qquad (E_2)$$

where

$$f_n = \frac{2}{l} \int_0^{x_0-\Delta x} (0) \sin \frac{n\pi x}{l}\, dx + \frac{2}{l} \int_{x_0-\Delta x}^{x_0+\Delta x} \frac{F_0}{2\Delta x} \sin \frac{n\pi x}{l}\, dx$$

$$+ \frac{2}{l} \int_{x_0+\Delta x}^{l} (0) \sin \frac{n\pi x}{l}\, dx = \frac{F_0}{l\,\Delta x} \int_{x_0-\Delta x}^{x_0+\Delta x} \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2F_0}{l} \sin \frac{n\pi x_0}{l} \; \frac{\sin\left(\frac{n\pi \Delta x}{l}\right)}{\left(\frac{n\pi\Delta x}{l}\right)} \qquad (E_3)$$

As $\Delta x \to 0$, (E_3) becomes $\quad f_n = \frac{2F_0}{l} \sin \frac{n\pi x_0}{l} \qquad (E_4)$

$$\therefore f(x) = \frac{2F_0}{l} \sum_{n=1}^{\infty} \sin \frac{n\pi x_0}{l} \sin \frac{n\pi x}{l} \qquad (E_5)$$

Since F_0 is moving along the bridge with constant velocity v, at time t, F_0 will be at a distance $x_0 = vt$ from left support. Hence the load distribution of Eq. (E_5) can be rewritten as

$$f(x,t) = \frac{2F_0}{l} \left\{ \sin \frac{\pi v t}{l} \sin \frac{\pi x}{l} + \sin \frac{2\pi v t}{l} \sin \frac{2\pi x}{l} + \cdots \right\} \qquad (E_6)$$

Setting $\omega = \frac{\pi v}{l}$, (E_6) can be expressed as

$$f(x,t) = \frac{2F_0}{l} \left\{ \sin \frac{\pi x}{l} \sin \omega t + \sin \frac{2\pi x}{l} \sin 2\omega t + \cdots \right\} \qquad (E_7)$$

Equation of motion, (E_1), becomes

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = f(x,t) = \sum_{n=1}^{\infty} F_n \sin \frac{n\pi x}{\ell} \sin n\omega t \qquad (E_8)$$

where $\quad F_n = \dfrac{2 F_0}{\ell} \quad ; \quad n = 1, 2, \ldots$

We can find the solution of Eq. (E_8) by superposing the solutions of the individual harmonic components. Thus, we need to find, for the n^{th} harmonic, the solution of the following equation:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = F_n \sin \frac{n\pi x}{\ell} \sin n\omega t \qquad (E_9)$$

The solution (particular integral) of Eq. (E_9) can be taken as

$$w(x,t) = w_n \sin \frac{n\pi x}{\ell} \sin n\omega t \qquad (E_{10})$$

Substitution of (E_{10}) into (E_9) gives

$$w_n = \frac{F_n \ell^4}{EI (n\pi)^4 \left\{ 1 - \left(\frac{n\omega}{\omega_n} \right)^2 \right\}} \qquad (E_{11})$$

where $\quad \omega_n = n^2 \pi^2 \sqrt{\dfrac{EI}{\rho A \ell^4}} \qquad (E_{12})$

Adding the homogeneous solution to the particular integral of Eq. (E_{10}), the general solution can be expressed as

$$w(x,t) = \sum_{i=1}^{\infty} \sin \frac{i\pi x}{\ell} \left\{ A_i \cos \omega_i t + B_i \sin \omega_i t \right\} + w_n \sin \frac{n\pi x}{\ell} \sin n\omega t \qquad (E_{13})$$

The initial conditions are given by

$$w(x,0) = 0 \quad , \quad \frac{\partial w}{\partial t}(x,0) = 0 \qquad (E_{14})$$

These initial conditions are satisfied if

$A_i = 0$ for all i, $\quad B_i = 0$ for all $i \neq n$

$B_n = -w_n \left(\dfrac{n\omega}{\omega_n} \right)$

$$w(x,t) = \frac{F_n \ell^4}{EI (n\pi)^4 \left\{ 1 - \left(\frac{n\omega}{\omega_n} \right)^2 \right\}} \sin \frac{n\pi x}{\ell} \left\{ \sin n\omega t - \frac{n\omega}{\omega_n} \sin \omega_n t \right\} \qquad (E_{15})$$

Since $\omega_n = n^2 \omega_1$, where ω_1 is the fundamental frequency of the bridge, Eq. (E_{15}) becomes

$$w(x,t) = \frac{F_n \, l^4}{EI\, \pi^4 \left\{ n^4 - \left(\frac{n\omega}{\omega_1}\right)^2 \right\}} \sin \frac{n\pi x}{l} \left\{ \sin n\omega t - \frac{n\omega}{n^2 \omega_1} \sin n^2 \omega_1 t \right\} \qquad (E_{16})$$

Thus the total response of the bridge can be expressed as

$$w(x,t) = \frac{2 F_0 \, l^3}{EI\, \pi^4} \left[\frac{\sin \frac{\pi x}{l}}{1 - \left(\frac{\omega}{\omega_1}\right)^2} \cdot \left\{ \sin \omega t - \left(\frac{\omega}{\omega_1}\right) \sin \omega_1 t \right\} \right.$$

$$+ \frac{\sin \frac{2\pi x}{l}}{2^4 - \left(\frac{2\omega}{\omega_1}\right)^2} \left\{ \sin 2\omega t - \left(\frac{\omega}{2\omega_1}\right) \sin 4\omega_1 t \right\}$$

$$\left. + \frac{\sin \frac{3\pi x}{l}}{3^4 - \left(\frac{3\omega}{\omega_1}\right)^2} \left\{ \sin 3\omega t - \left(\frac{\omega}{3\omega_1}\right) \sin 9\omega_1 t \right\} + \cdots \right] \qquad (E_{17})$$

Chapter 10

Vibration Measurement and Applications

10.1 Voltage sensitivity $= v = 0.098$ volt-meter/Newton

thickness $= t = 2$ mm $= \frac{2}{1000}$ m

output voltage $= 220$ volts , pressure applied $= p_x = ?$

$$E = v\, t\, p_x \quad \Rightarrow \quad 220 = (0.098)\left(\frac{2}{1000}\right) p_x$$

$$\therefore \; p_x = 1.1224 \times 10^6 \; N/m^2$$

10.2 $m = 0.5$ kg , $k = 10000$ N/m , $c \simeq 0$

amplitude $= Y = 4/1000$ m

total displacement of mass $= x = 12/1000$ m

relative displacement $= z = x - y = 8/1000$ m

$$Z = \frac{r^2 Y}{1-r^2} \qquad \text{i.e.,} \qquad \frac{8}{1000} = \left\{ \frac{(4/1000)\, r^2}{1 - r^2} \right\} \Rightarrow r^2 = 2/3$$

$$r = \frac{\omega}{\omega_n} = 0.8165$$

$$\omega = r\,\omega_n = 0.8165 \sqrt{\frac{10000}{0.5}} = 115.4705 \text{ rad/sec} = 18.3777 \text{ Hz}$$

10.3
$$Z = \frac{r^2 Y}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad \text{where} \quad \zeta = \frac{c}{c_{cri}} = \frac{1}{\sqrt{2}}$$

$$= \frac{r^2 Y}{\sqrt{(1-r^2)^2 + (\sqrt{2}\, r)^2}} = \frac{r^2 Y}{\sqrt{1 + r^4}}$$

10.4 speed range:

500 rpm $= 52.36$ rad/sec $-$ 1500 rpm $= 157.08$ rad/sec

$$x = X \cos \omega t \; ; \qquad Z = \frac{r^2 Y}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad (E_1)$$

CASE (i): Let $\zeta = 0$

Eq. (E_1) gives, for 2% error,

$$\frac{Z}{Y} = \frac{r^2}{|r^2 - 1|} = 1.02 \quad \Rightarrow \quad r = \frac{\omega}{\omega_n} = 7.1414$$

$$\omega_n = \frac{\omega}{7.1414} = \frac{500}{7.1414} \text{ or } \frac{1500}{7.1414} = 70.0143 \text{ or } 210.0428 \text{ rpm}$$

$$= 1.1669 \text{ or } 3.5007 \text{ Hz} = 7.3319 \text{ or } 21.9957 \text{ rad/sec}$$

$\therefore \omega_n = 1.1669$ Hz

CASE (ii) : Let $\zeta = 0.6$

Eq. (E_1) gives $$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = E = 1.02$$

$$\Rightarrow \quad 0.0404\, r^4 - 0.5826\, r^2 + 1.0404 = 0$$

which gives $r = 2.2562,\ 3.0546$

Since the quantity E attains maximum at

$$r = \frac{1}{\sqrt{1-2\zeta^2}} = 1.8898 \quad \text{for } \zeta = 0.6$$

we use $r = 2.2562$.

$\therefore$ Maximum $\omega_n = \omega/r = 500/(2.2562 \times 60) = 3.6935$ Hz

---

(10.5) Error factor for vibrometer is $$E = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Maximum of E occurs at $$r^* = \frac{1}{\sqrt{1-2\zeta^2}}$$

For $\zeta = 0$, $E = \dfrac{r^2}{|1-r^2|}$ and $r^* = 1$

Since the range is $4 \le r < \infty$, we use $r = 4$ for maximum error.

$$E\Big|_{r=4} = \frac{4^2}{|1-4^2|} = 1.0667$$

Percent error $= (E-1)\,100 = 6.67\%$

(10.6) Error factor $= E = \dfrac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

E attains maximum at $r^* = 1/\sqrt{1-2\zeta^2}$

For $\zeta = 0.67$, $r^* = 3.1281$ and $E\Big|_{r=r^*,\ \zeta=0.67} = 1.0053$

Percent error $= 0.53\%$

10.7 Select the vibrometer on the basis of lowest frequency being measured.

(a) $\zeta = 0$:

From Eq. (10.19), $\frac{Z}{Y} = 1.03 = \frac{r^2}{|r^2-1|}$, $r^2 = \frac{1.03}{0.03} = 34.3333$

$r = \frac{\omega}{\omega_n} = \frac{2\pi(500)}{60\ \omega_n} = 5.8595$

$\omega_n = 8.9359$ rad/sec $= 1.4222$ Hz

(b) $\zeta = 0.6$:

From Eq. (10.19), $\frac{Z}{Y} = 1.03 = \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = E$

$(1.03)^2\{1 + r^4 - 2r^2 + 4r^2\zeta^2\} = r^4$

i.e., $0.057404\ r^4 - 0.56\ r^2 + 1 = 0$

i.e., $r^2 = 2.3535\ ,\quad 7.4018$

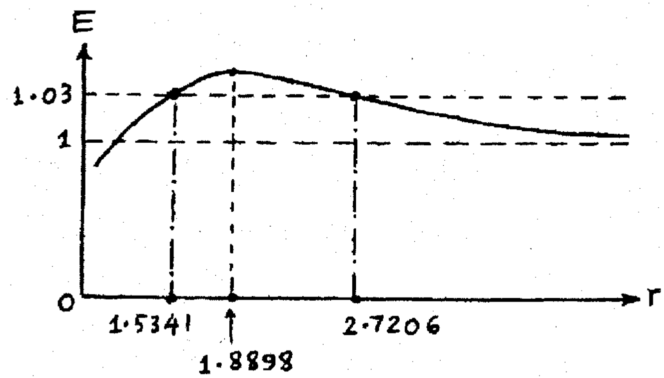
i.e., $r = 1.5341^{\dagger},\quad 2.7206$

By selecting $r = 2.7206$,

$\omega_n = \frac{1000\ \pi}{60\ (2.7206)}$

$= 19.2458$ rad/sec

$= 3.0630$ Hz



† The quantity E attains maximum at

$r = \frac{1}{\sqrt{1-2\zeta^2}} = 1.8898$

for $\zeta = 0.6$. Hence we have to take $r = 2.7206$ to avoid peak of E.

10.8 $\delta_{st} = \frac{10}{1000}$ m , $\omega = 4000$ rpm $= 418.88$ rad/sec

$\omega_n = \sqrt{g/\delta_{st}} = \sqrt{9.81\left(\frac{1000}{10}\right)} = 31.3209$ rad/sec

$r = \omega/\omega_n = 418.88/31.3209 = 13.3738$

Let $\zeta = 0$:

Error factor $= \left.\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\right|_{\zeta=0} = \frac{r^2}{|1-r^2|} = \frac{13.3738^2}{|1-13.3738^2|}$

$= 1.0056$

(i) Maximum displacement $= Y = Z/1.0056 = 1/1.0056 = 0.9944$ mm

(ii) Maximum velocity $= \omega Y = (418.88)(0.9944) = 416.5473$ mm/sec

(iii) Maximum acceleration $= \omega^2 Y = (418.88)^2(0.9944) = 174483.35$ mm/sec^2

10.9 $$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad ---- \quad (E_1)$$

Maximum of $\frac{Z}{Y}$ occurs when $r = r^* = \frac{1}{\sqrt{1-2\zeta^2}}$ (see Eq. (3.84))

For $\zeta = 0.5$, $r^* = \frac{1}{\sqrt{0.5}} = 1.4142$, $(r^*)^2 = 2$

$$\left.\frac{Z}{Y}\right|_{r^*} = \frac{2}{\sqrt{(1-2)^2 + (2\times 0.5 \times 1.4142)^2}} = \frac{2}{\sqrt{3}} = 1.1547$$

When error is one percent, $\frac{Z}{Y} = 1.01$ or $\frac{Y}{Z} = 0.9901$

Eq. (E_1) can be rewritten as:

$$\left|\frac{Y}{Z}\right|^2 = \frac{(1-r^2)^2 + 4\zeta^2 r^2}{r^4}$$

$$r^4\left(1 - \left|\frac{Y}{Z}\right|^2\right) - r^2\left(2 - 4\zeta^2\right) + 1 = 0 \quad ---- \quad (E_2)$$

For $\frac{Y}{Z} = 0.9901$ and $\zeta = 0.5$, (E_2) becomes

$$0.0197\ r^4 - r^2 + 1 = 0$$

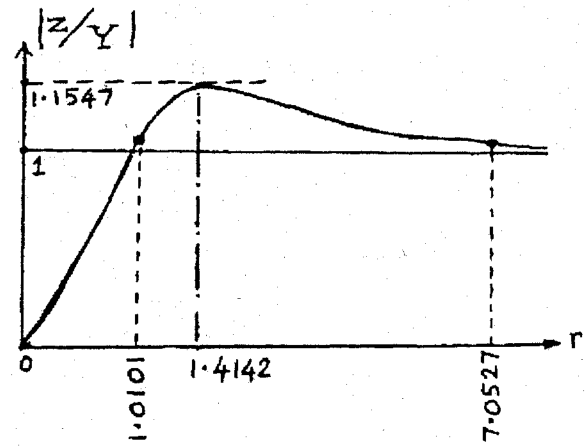
$$r^2 = 1.0203, \quad 49.7411$$

$$r = \frac{\omega}{\omega_n} = 1.0101, \quad 7.0527$$

Lowest frequency for one percent accuracy

$= 7.0527\,(5)$

$= 35.2635$ Hz



10.10 Frequency range > 100 Hz, maximum error $= 2\%$

$k = 4000$ N/m, $c = 0 \Rightarrow \zeta = 0$, $m = ?$

For vibrometer with $\zeta = 0$,
$$\frac{Z}{Y} = \left.\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\right|_{\zeta=0}$$
$$= \frac{r^2}{|1-r^2|} = 1.02$$

or $\quad r^2/(1-r^2) = -1.02$ since r must be greater than one for higher frequencies

or $\quad r^2 = 51$ i.e., $r = 7.1414$

Minimum impressed frequency $= \omega = 10$ Hz

Since $r = \omega/\omega_n = 100/\omega_n = 7.1414$, $\quad \omega_n = 14.0029$ Hz $= 87.9827 \frac{\text{rad}}{\text{sec}}$

$= \sqrt{k/m}$

$\therefore \quad m = k/\omega_n^2 = 4000/(87.9827)^2 = 0.5167$ kg

---

(10.11) $\omega_n = 10$ Hz, $\quad \omega_d = 8$ Hz $= \omega_n \sqrt{1-\zeta^2} = 10\sqrt{1-\zeta^2} \Rightarrow \zeta = 0.6$

Let the lowest frequency $= \omega_0 \Rightarrow r_0 = \omega_0/\omega_n$

Error $= 2\%$ in the range $r_0 \le r < \infty$

$$\text{Error} = E = 1.02 = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\times 0.6\ r)^2}}$$

$$\Rightarrow \quad 0.0404\ r^4 - 0.5826\ r^2 + 1.0404 = 0$$

$$\Rightarrow \quad r = 2.2562,\ 3.0546$$

Let $r_0 = 2.2562$:

$$\omega_0 = r_0\,\omega_n = 2.2562\,(10) = 22.562 \text{ Hz}$$

Let $r_0 = 3.0546$:

$$\omega_0 = r_0\,\omega_n = 3.0546\,(10) = 30.546 \text{ Hz}$$

$\therefore$ Lowest frequency $= 22.562$ Hz $= 141.7616$ rad/sec

---

(10.12) Error factor for accelerometer $= E = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

Maximum of E occurs at $r^* = \sqrt{1-2\zeta^2}$

When $\zeta = 0$, $\quad r^* = 1$ and $\quad E = \dfrac{1}{|1-r^2|}$

Since the range is $0 \le r \le 0.65$, we use $r = 0.65$:

$$E\Big|_{r=0.65} = \frac{1}{|1-0.65^2|} = 1.7316$$

Percent error $= (E-1)\,100 = 73.16\%$

(10.13) Error factor $= E = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

Value of r at which E attains maximum is $\quad r^* = \sqrt{1-2\zeta^2}$

When $\zeta = 0.75$, $\quad E = \dfrac{1}{\sqrt{(1-r^2)^2 + 2.25\,r^2}}$ $\quad$ and $\quad r^* = \sqrt{1-1.125}$

$= \text{imaginary}$

Since the range is $0 \le r \le 0.6$, we use $r = 0.6$

$$E\Big|_{r=0.6} = \frac{1}{\sqrt{(1-0.36)^2 + 2.25\,(0.6)^2}} = 0.9055$$

Percent error $= (E-1)\,100 = -9.45\%$

10.14 $m = 0.05$ kg, max error = 3% over frequency range of 0 to 100 Hz

Find k and c.

For accelerometer, error factor $= E = \dfrac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$

E attains maximum at $r = r^* = \sqrt{1-2\zeta^2}$

and $E_{max} = E\big|_{r^*} = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$

(i) Consideration of maximum error:

let error $= e = E-1 = 0.03 = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} - 1$

upon rearrangement, this leads to $\zeta^4 - \zeta^2 + 0.23565 = 0$

or $\zeta = 0.6164,\ 0.7874$

r^* at $\zeta = 0.6164 = \sqrt{1-2(0.6164)^2} = 0.49$

r^* at $\zeta = 0.7874 = \sqrt{1-2(0.7874)^2} =$ imaginary

(ii) Consideration of minimum error:

let error $= e = E-1 = -0.03 = \dfrac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} - 1 \qquad (E_1)$

with $\zeta = 0.6164$, Eq. (E_1) can be simplified as

$r^4 - 0.4802\, r^2 - 0.0628 = 0$

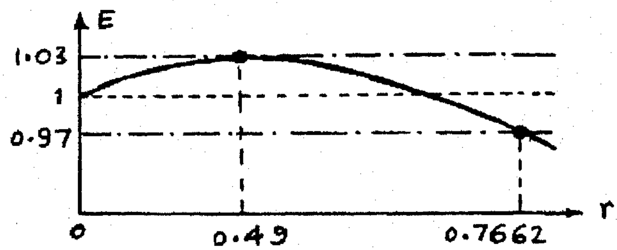
$\Rightarrow r^2 = 0.5871,\ -0.1069$ or $r = 0.7662$

At the maximum frequency,

$\omega = 2\pi(100) = 628.32$ rad/sec

$\omega_n = \omega/r = 628.32/0.7662$

$= 820.0470$ rad/sec



$k = m\,\omega_n^2 = 0.05\,(820.047)^2$

$= 33623.8541$ N/m

$c_c = 2m\,\omega_n = \dfrac{c}{\zeta} \Rightarrow c = 2m\,\omega_n\,\zeta = 2(0.05)(820.047)(0.6164)$

$= 50.5477$ N-s/m

10.15 $m = 0.1$ kg, $k = 10000$ N/m, $c = 0 \Rightarrow \zeta = 0$

$\omega_n = \sqrt{k/m} = \sqrt{10000/0.1} = 316.2278$ rad/sec

Engine speed $= \omega = 1000$ rpm $= 104.72$ rad/sec

$r = \omega/\omega_n = 104.72/316.2278 = 0.3312$

peak-to-peak travel of mass = 10 mm

Find: Y, ωY, $\omega^2 Y$.

We have, from Eq. (10.19),

$$\left.\frac{Z}{Y}\right|_{\zeta=0} = \frac{r^2}{|1-r^2|} = \frac{0.3312^2}{|1-0.3312^2|} = 0.1232$$

Since peak-to-peak travel of mass = 10 mm, Z = 5 mm

$$Y = Z/0.1232 = 5/0.1232 = 40.5844 \text{ mm}$$

Max displacement of foundation = Y = 40.5844 mm

Max velocity of foundation = ωY = 4249.9984 mm/sec

Max acceleration of foundation = $\omega^2 Y$ = 445059.8291 mm/sec^2

10.16 Maximum speed = 3000 rpm = 50 Hz ; $r = \frac{\omega}{\omega_n} = \frac{50}{100} = 0.5$

For accelerometer, $$\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = 1 + \text{error} = 0.9 \qquad (E_1)$$

Here c = 20 N-s/m ; $\zeta = \frac{c}{2m\omega_n} = \frac{20}{2m\,(100\times 2\pi)} = \frac{0.015915}{m}$

For r = 0.5, Eq. (E_1) gives $\zeta = 0.8198$

$$c_c = \frac{c}{\zeta} = 20/0.8198 = 24.3962 \text{ N-s/m}$$

$$m = \frac{c_c}{2\omega_n} = \frac{24.3962}{2\,(100\times 2\pi)} = 0.01941 \text{ kg} = 19.41 \text{ grams}$$

$$k = m\,\omega_n^2 = 0.01941\,(100\times 2\pi)^2 = 7622.7967 \text{ N/m}$$

10.17 $\omega_n = 2\pi\,(0.5) = \pi$ rad/sec ; $\omega_d = \omega_n \sqrt{1-\zeta^2}$

$$\sqrt{1-\zeta^2} = \omega_d/\omega_n = 0.48/0.5 \Rightarrow \zeta = 0.28$$

$r_1 = \omega_1/\omega_n = 4\pi/\pi = 4$; $r_2 = \omega_2/\omega_n = 8$; $r_3 = \omega_3/\omega_n = 12$

$$\phi_i = \tan^{-1}\left(\frac{2\zeta r_i}{1-r_i^2}\right)$$

$$\phi_1 = \tan^{-1}\left(\frac{2\,(0.28)\,4}{1-16}\right) = -8.4934°$$

$$\phi_2 = \tan^{-1}\left(\frac{2\times 0.28\times 8}{1-64}\right) = -4.0675°$$

$$\phi_3 = \tan^{-1}\left(\frac{2\times 0.28\times 12}{1-144}\right) = -2.6905°$$

$$\frac{r_1^2}{\sqrt{(1-r_1^2)^2 + (2\zeta r_1)^2}} \times 20 = 21.0994$$

$$\frac{r_2^2}{\sqrt{(1-r_2^2)^2 + (2\zeta r_2)^2}} \times 10 = 10.1331$$

$$\frac{r_3^2}{\sqrt{(1-r_3^2)^2 + (2\zeta r_3)^2}} \times 5 = 5.0294$$

Record indicated by vibrometer is given by

$$z(t) = 21.0994 \sin(4\pi t + 8.4934^\circ) + 10.1331 \sin(8\pi t + 4.0675^\circ) + 5.0294 \sin(12\pi t + 2.6905^\circ) \quad \text{mm}$$

10.18

$$x(t) = 20 \sin 50t + 5 \sin 150t \quad \text{mm} \qquad (E_1)$$

$$\ddot{x}(t) = -20(50)^2 \sin 50t - 5(150)^2 \sin 150t \quad \text{mm/sec}^2$$

$$= -50\,000 \sin 50t - 112\,500 \sin 150t \quad \text{mm/sec}^2 \qquad (E_2)$$

$\omega_n = 100$ rad/sec , $\omega_d = \omega_n \sqrt{1-\zeta^2} = 80 \Rightarrow \zeta = 0.6$

$$r_1 = \frac{\omega_1}{\omega_n} = \frac{50}{100} = 0.5 \; ; \qquad r_2 = \frac{\omega_2}{\omega_n} = \frac{150}{100} = 1.5$$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r_1}{1-r_1^2}\right) = \tan^{-1}\left(\frac{2 \times 0.6 \times 0.5}{1-0.25}\right) = 38.6598^\circ$$

$$\phi_2 = \tan^{-1}\left(\frac{2\zeta r_2}{1-r_2^2}\right) = \tan^{-1}\left(\frac{2 \times 0.6 \times 1.5}{1-2.25}\right) = -55.2222^\circ$$

$$\frac{50\,000}{\sqrt{(1-r_1^2)^2 + (2\zeta r_1)^2}} = 52\,057.9206$$

$$\frac{112\,500}{\sqrt{(1-r_2^2)^2 + (2\zeta r_2)^2}} = 51\,335.6229$$

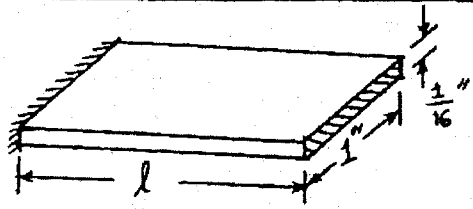
output of the accelerometer is given by

$$\ddot{z}(t) = -52\,057.9206 \sin(50t - 38.6598^\circ) - 51\,335.6229 \sin(150t + 55.2222^\circ) \quad \text{mm/sec}^2 \qquad (E_3)$$

It can be seen that Eq. (E_3) is substantially different from Eq. (E_2).

10.19

For given beam,



$A = 1\left(\frac{1}{16}\right) = 0.0625 \text{ in}^2$

$I = \frac{1}{12}(1)\left(\frac{1}{16}\right)^3 = 20.35 \times 10^{-6} \text{ in}^4$

$l = 2''$ to $10''$

For a cantilever beam, Fig. 8.15 gives

$$\omega_n = (\beta_n l)^2 \left(\frac{EI}{\rho A l^4} \right)^{1/2}$$

where $(\beta_1 l)^2 = (1.875104)^2 = 3.516015$

$(\beta_2 l)^2 = (4.694091)^2 = 22.03449$

$(\beta_3 l)^2 = (7.854757)^2 = 61.69721$

$(\beta_4 l)^2 = (10.995541)^2 = 120.90192$

For spring steel, $E = 30 \times 10^6$ psi , $\rho g = 0.283$ lb/in^3

$$\left(\frac{EI}{\rho A l^4} \right)^{\frac{1}{2}} = \left\{ \frac{30\times 10^6 \; (20.35 \times 10^{-6})}{\left(\frac{0.283}{386.4}\right) (0.0625) \; l^4} \right\}^{1/2} = \frac{3652.0}{l^2}$$

$$\omega_n = (\beta_n l)^2 \left(\frac{3652.0}{l^2} \right)$$

The first four frequencies are given below:

	l^2	ω_1	ω_2	ω_3	ω_4
$l = 2''$	4	3210.1020	20117.4894	56329.5527	110383.4530
$l = 10''$	100	128.4049	804.6996	2253.1821	4415.3381

Hence the range of frequencies that can be measured is given by $\omega > 128.4049$ rad/sec.

However, for first mode only (which is easiest to excite), the range of frequencies is $128.4049 \frac{\text{rad}}{\text{sec}} \le \omega \le 3210.102 \frac{\text{rad}}{\text{sec}}$.

10.20

$$\frac{k X_R}{F_0} = \frac{1 - r^2}{(1-r^2)^2 + (2\zeta r)^2} = \frac{N}{D} \text{ (assume)}$$

$$\frac{d}{dr}\left(\frac{k X_R}{F_0} \right) = \frac{D \frac{dN}{dr} - N \frac{dD}{dr}}{D^2} = 0 \quad \text{i.e.,} \quad D \frac{dN}{dr} - N \frac{dD}{dr} = 0$$

$$\text{or} \quad \left\{ (1-r^2)^2 + (2\zeta r)^2 \right\} (-2r) - (1-r^2) \left\{ 2(1-r^2)(-2r) + 2(2\zeta r)(2\zeta) \right\} = 0$$

This equation can be simplified as

$$r^4 - 2r^2 + (1 - 4\zeta^2) = 0$$

and its solution is given by

$$r^2 = 1 \pm 2\zeta \quad \text{or} \quad r = \sqrt{1+2\zeta} \; ; \; \sqrt{1-2\zeta}$$

Since

$$\frac{k X_R}{F_0} \Big|_{r = \sqrt{1+2\zeta}} = - \frac{1}{4\zeta(1+\zeta)} \tag{1}$$

and

$$\frac{k\,X_R}{F_0}\Big|_{r=\sqrt{1-2\varsigma}} = \frac{1}{4\,\varsigma\,(1-\varsigma)} \tag{2}$$

we note that $r = R_1 = \sqrt{1-2\,\varsigma}$ corresponds to a maximum and $r = R_2 = \sqrt{1+2\,\varsigma}$ corresponds to a minimum of X_R.

10.21

$$\frac{k\,X_I}{F_0} = \frac{-2\,\varsigma\,r}{(1-r^2)^2 + 4\,\varsigma^2\,r^2}$$

$$\frac{d}{dr}\left(\frac{k\,X_I}{F_0}\right) = \frac{D\,\frac{dN}{dr} - N\,\frac{dD}{dr}}{D^2} = 0 \tag{1}$$

where $\frac{dN}{dr} = -2\,\varsigma$ and $\frac{dD}{dr} = -2\,(1-r^2)\,(2\,r) + 8\,\varsigma^2\,r$

By setting the numerator of Eq. (1) equal to zero, we obtain

$$\left\{1 + r^4 - 2\,r^2 + 4\,\varsigma^2\,r^2\right\}(-2\,\varsigma) - (-2\,\varsigma\,r)\left\{-4\,r + 4\,r^3 + 8\,\varsigma^2\,r\right\} = 0$$

which can be simplified to obtain

$$3\,r^4 + (4\,\varsigma^2 - 2)\,r^2 - 1 = 0 \tag{2}$$

The roots of Eq. (2) are given by

$$r^2 = \frac{1 - 2\,\varsigma^2 - 2\,\sqrt{\varsigma^4 - \varsigma^2 + 1}}{3}\ ;\ \frac{1 - 2\,\varsigma^2 + 2\,\sqrt{\varsigma^4 - \varsigma^2 + 1}}{3} \tag{3}$$

Since it is difficult to determine, from Eq. (3), the correct value of r that corresponds to the minimum of X_I, we use a numerical computation. For $\varsigma = 0.1$, for example, Eq. (3) gives

$$r^2 = -0.0030\ ;\ 0.6583$$

This shows that

$$r = \left\{\frac{1 - 2\,\varsigma^2 + 2\,\sqrt{\varsigma^4 - \varsigma^2 + 1}}{3}\right\}^{\frac{1}{2}} \tag{4}$$

corresponds to the minimum of X_I. For small values of ς, $\varsigma^2 \ll 1$ and Eq. (4) gives

$$r \approx 1 \tag{5}$$

Thus X_I attains its minimum value close to $r = 1$.

10.22 Response of a single d.o.f. system with hysteretic damping is given by Eq. (3.106):

$$\frac{X}{F_0} = \frac{1}{k - m\,\omega^2 + i\,k\,\beta}$$

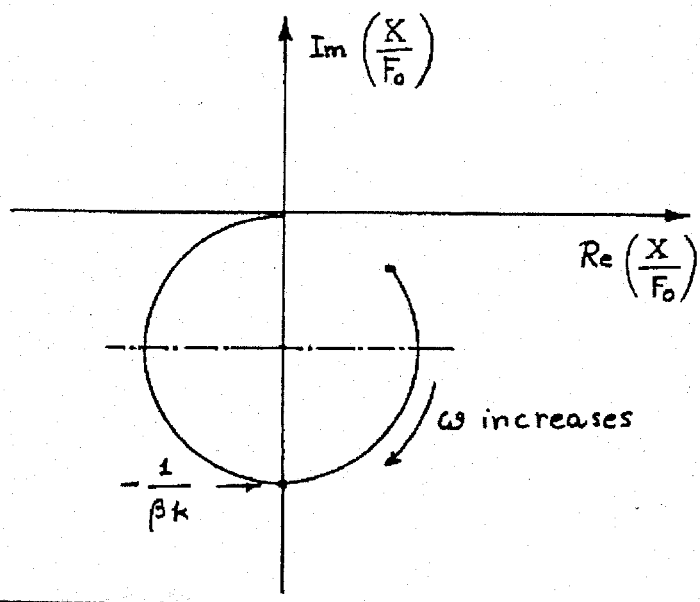
$$\mathrm{Re}\left(\frac{X}{F_0}\right) = \frac{k - m\,\omega^2}{(k - m\,\omega^2)^2 + (k\,\beta)^2} \quad . \quad \mathrm{Im}\left(\frac{X}{F_0}\right) = \frac{-k\,\beta}{(k - m\,\omega^2)^2 + (k\,\beta)^2}$$

$$\left[\mathrm{Re}\left(\frac{X}{F_0}\right)\right]^2 + \left[\mathrm{Im}\left(\frac{X}{F_0}\right)\right]^2 = \frac{1}{(k - m\,\omega^2)^2 + (k\,\beta)^2} \tag{1}$$

It can be verified that Eq. (1) can be rewritten as

$$\left[\mathrm{Re}\left(\frac{X}{F_0}\right)\right]^2 + \left[\mathrm{Im}\left(\frac{X}{F_0}\right) + \frac{1}{2\,\beta\,k}\right]^2 = \left(\frac{1}{2\,\beta\,k}\right)^2 \tag{2}$$

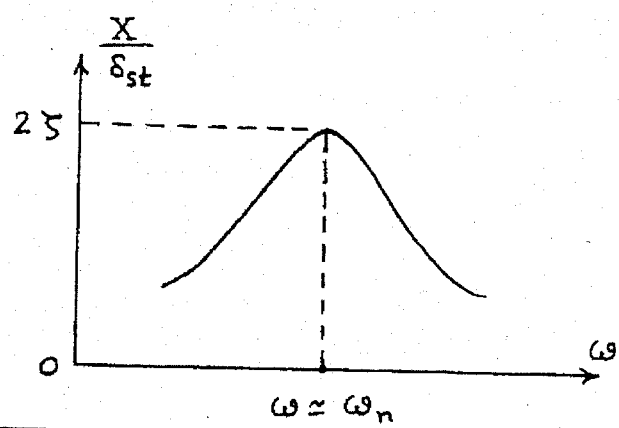
Eq. (2) shows that the locus of $\left(\frac{X}{F_0}\right)$ as ω increases from zero is part of a circle, with center $\left(0, -\frac{1}{2\,k\,\beta}\right)$ and radius $\frac{1}{2\,k\,\beta}$, as shown in the following figure.



(10.23) The peak of Bode diagram is equal to $\approx \frac{1}{2\,\varsigma}$. In the present case, peak-to-peak value is plotted; hence $X \approx \frac{0.45}{2}$ mil $= 0.225$ mil.

$$\frac{X}{\delta_{st}} = \frac{0.225}{0.05} = 4.5 = \frac{1}{2\,\varsigma}$$

or $\varsigma = 0.1111$.

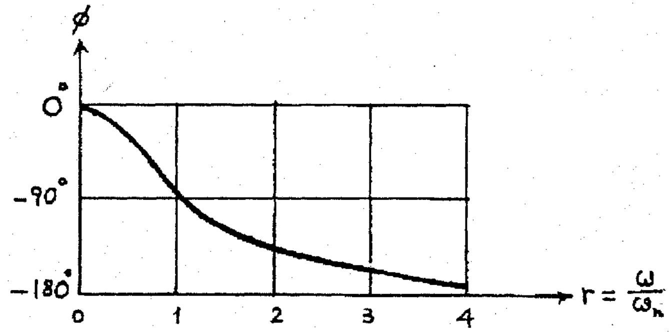


(10.24) Reduction in amplitude from 6.8 in/sec to 0.8 in/sec in 7 cycles or 22 milliseconds. Eq. (2.92) gives:

$$\frac{1}{7}\ \ln\left(\frac{x_1}{x_8}\right) = \delta \quad \text{or} \quad \delta = \frac{1}{7}\ \ln\left(\frac{6.8}{0.8}\right) = 0.3057$$

Hence $\varsigma = \dfrac{\delta}{2\ \pi} = 0.04866.$

(10.25) Typical Bode plot of phase angle is shown in the figure.



(a) $\phi = 90°$ at $r = \dfrac{\omega}{\omega_n} \approx 1$. Hence the value of ω_n can be determined from the value of r corresponding to $\phi = 90°$.

(b) Since

$$\phi = \tan^{-1}\left(-\frac{2\ \varsigma\ r}{1 - r^2}\right) = \tan^{-1}\left(-\frac{c\ \omega}{k - m\ \omega^2}\right)$$

we find

$$-\frac{c\ \omega_1}{k - m\ \omega_1^2} = -1 \quad \text{and} \quad -\frac{c\ \omega_2}{k - m\ \omega_2^2} = 1$$

where ω_1 and ω_2 correspond to the half power points. Hence, by finding the values of ω corresponding to $\phi = -45°$ and $\phi = -135°$, we obtain ω_1 and ω_2. From these values, the damping ratio can be found using the relation:

$$\varsigma = \frac{\omega_2 - \omega_1}{2\ \omega_n}$$

10.26 10.27 10.28

Characteristic	Problem 10.26	Problem 10.27	Problem 10.28
n	16	18	18
N	750 rpm	1000 rpm	1500 rpm
d	15 mm	2 cm	10 mm
D	100 mm	15 cm	80 mm
α	30°	20°	40°
$\frac{d}{D}\cos\alpha$	0.1299	0.1253	0.09576
Dominant frequency of vibration			
Inner race defect	6779.4 cycles/min (1078.97 Hz)	10127.7 cycles/min (1611.87 Hz)	14792.8 cycles/min (2354.33 Hz)
Outer race defect	5220.6 cycles/min (830.88 Hz)	7872.3 cycles/min (1252.91 Hz)	12207.2 cycles/min (1942.84 Hz)
Ball or roller defect	1214.6 cycles/min (193.31 Hz)	1761.7 cycles/min (280.39 Hz)	2188.1 cycles/min (348.25 Hz)
Cage defect	326.3 cycles/min (51.93 Hz)	437.3 cycles/min (69.61 Hz)	678.2 cycles/min (107.93 Hz)

10.29 $f(x) = \frac{1}{4} \quad ; \quad 1 \le x \le 5$

$$\bar{x} = \text{mean value of } x = \int_1^5 f(x)\, x\, dx = \frac{1}{4}\left(\frac{x^2}{2}\right)_1^5 = 3$$

$$\sigma^2 = (\text{standard deviation})^2 = \int_1^5 (x - \bar{x})^2 f(x)\, dx$$

$$= \frac{1}{4}\int_1^5 (x-3)^2\, dx = \frac{1}{4}\left(\frac{x^3}{3} - 3x^2 + 9x\right)_1^5 = \frac{4}{3}$$

$$k = \text{kurtosis} = \frac{1}{\sigma^4}\int_{-\infty}^{\infty} (x - \bar{x})^4 f(x)\, dx = \frac{9}{16}\int_1^5 (x-3)^4 \left(\frac{1}{4}\right) dx$$

Let $y = x - 3$ so that $dy = dx$. This gives

$$k = \frac{9}{64}\int_{y=-2}^{2} y^4\, dy = \frac{9}{5}$$

10.30 $\bar{x}$ = mean value of x = $\sum_i f(x_i)\, x_i$

$$= \frac{1}{32}(1) + \frac{3}{32}(2) + \frac{3}{16}(3) + \frac{6}{16}(4) + \frac{3}{16}(5) + \frac{3}{32}(6) + \frac{1}{32}(7) = 4$$

$$\sigma^2 = (\text{standard deviation})^2 = \sum_i (x_i - \bar{x})^2 f(x_i)$$

$$= (1-4)^2 \frac{1}{32} + (2-4)^2 \frac{3}{32} + (3-4)^2 \frac{3}{16} + (4-4)^2 \frac{6}{16}$$

$$+ (5-4)^2 \frac{3}{16} + (6-4)^2 \frac{3}{32} + (7-4)^2 \frac{1}{32} = \frac{27}{16} = 1.6875$$

$$\sum_i (x_i - \bar{x})^4 f(x_i) = (1-4)^4 \frac{1}{32} + (2-4)^4 \frac{3}{32} + (3-4)^4 \frac{3}{16}$$

$$+ (4-4)^4 \frac{6}{16} + (5-4)^4 \frac{3}{16} + (6-4)^4 \frac{3}{32} + (7-4)^4 \frac{1}{32} = \frac{135}{16} = 8.4375$$

$$k = \text{kurtosis} = \frac{1}{\sigma^4} \sum_i (x_i - \bar{x})^4 f(x_i) = \frac{8.4375}{1.6875^2} = 2.9630$$

10.31 Range of ω = 62.832 to 314.16 rad/sec = 600 to 3000 rpm
Max acceleration level = 10 g = 98.1 m/sec^2
Max weight of specimen = 10 N
Max vibration amplitude = 0.0025 m
Frequency range:
Variable speed electric motor can be used to obtain the frequency range (for a mechanical shaker).
Vibration amplitude:
If $y(t) = A \sin \omega t$, (E₁)
acceleration = $A\omega^2$.
At $\omega = 314.16$ rad/sec, amplitude needed to achieve the maximum acceleration is:

$$\text{amplitude } (A) = \text{acceleration}/\omega^2$$

$$= 98.1/(314.16)^2 = 0.9939 \times 10^{-3} \text{ m}$$

At $\omega = 62.832$ rad/sec, amplitude needed to achieve the maximum acceleration is:

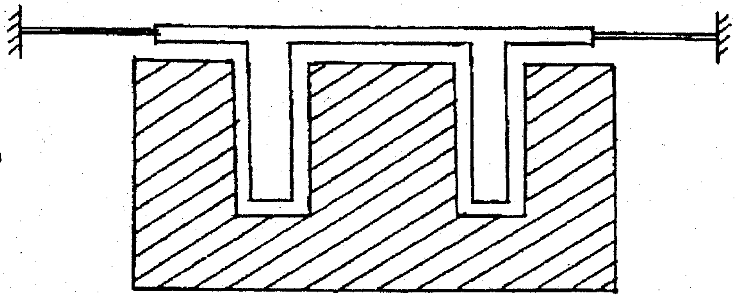
amplitude (A) = acceleration $/\omega^2$

$= 98.1/(62.832)^2 = 0.02485$ m

(amplitude is too high ; hence direct application of $y(t)$ is not permitted).

Mechanical shaker of the type shown in Fig. 10.16 can be used. Electrodynamic shaker of the type shown in Fig. 10.17 (a) can also be used.

Electromagnetic shaker:



Max force (F_{max}) available depends on:

(a) magnetic field strength

(b) number of turns

(c) coil diameter

(d) current flowing

Limitations are: (a) material strength, (b) cooling provided.

Max acceleration $= \left(\dfrac{F_{max}}{m_s + m_t} \right)$

where m_s = mass of specimen and m_t = mass of shaker table.

(10.32) Speed range: 300 - 600 rpm

Frequency range: 31.416 - 62.832 rad/sec

Number of reeds = 12

Uniform spacing of frequencies give the reed frequencies as:

$\Omega_1, \ldots, \Omega_{12}$ = 31.416, 34.272, 37.128, 39.984, 42.840,
45.696, 48.552, 51.408, 54.264, 57.120, 59.976, 62.832 rad/sec

Let each reed be considered as a cantilever beam of cross section a x b inches. Let lengths of all reeds be same and the material be aluminum for light weight. The fundamental natural frequency of a reed is given by (Fig. 8.15):

$$\omega_1 = (\beta_1\, \ell)^2 \left\{ \frac{EI}{\rho\, A\, \ell^4} \right\}^{\frac{1}{2}} = (1.8751^2) \left\{ \frac{EI}{\rho\, A\, \ell^4} \right\}^{\frac{1}{2}}$$

$$= 3.516 \left\{ \frac{(10^7)\, I}{(0.1/386.4)\, A\, \ell^4} \right\}^{\frac{1}{2}} = 69.1142\,(10^4) \left\{ \frac{I}{A\, \ell^4} \right\}^{\frac{1}{2}} \qquad (1)$$

By equating ω_1 given by Eq. (1) to $\Omega_1, \ldots, \Omega_{12}$, in turn, the proper value of $\left\{ \dfrac{I}{A\, \ell^4} \right\}^{\frac{1}{2}}$

needed for different reeds can be computed. By selecting a common value of ℓ for all reeds, the cross section of any reed can then be found to achieve the required value of $\left\{\frac{I}{A\,\ell^4}\right\}^{\frac{1}{2}}$.

10.33 Iterative process is to be used.

1. Select trial values of the design parameters (material of the beam and its dimensions).

2. Model the beam as a spring-mass system with:
   m = end mass = 50% of mass of beam:

$$m = \frac{1}{2}\,\rho\,A\,\ell \tag{1}$$

   k = stiffness of a cantilever beam:

$$k = \frac{3\,E\,I}{\ell^3} \tag{2}$$

3. Equations of motion:

$$m\,\ddot{x} + k\,(x - y) = 0$$

$$\text{or} \quad m\,\ddot{z} + k\,z = -\,m\,\ddot{y} \tag{3}$$

   where z = relative displacement of end mass.

4. Since $\ddot{y}_{max} = 0.2$ g, assume a constant force of - m (0.2 g) on the right hand side of Eq. (3) and solve the equation to find z(t).

5. From the known z_{max} value, compute the maximum stress (σ_{max}) induced in the beam. If σ_{max} is less than the yield stress of the material, the design is complete. Otherwise, go to step 1 and change one or more design parameters and repeat the procedure until a satisfactory design is found.